Understanding Central Bank Loss Functions:

Implied and Delegated Targets

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Abstract: The paper studies the dynamic nature of optimal solutions under commitment in Barro-Gordon and new-Keynesian models and, finds two interesting parameters — the implied targets and the persistence parameter that governs the adjustment toward the implied targets. The implied targets generally differ from the social ones, but exhibit a trade-off between targets and equal the long-run equilibrium values of target variables. The implied targets prove consistent with the models and the social targets do not. Moreover, the implied targets emerge in the long run according to the persistence parameter. As such, the government delegates to the central bank short-term, state-contingent targets, which guide discretionary policy to evolve along optimal paths as these targets converge to their long-run implied targets. For the Barro-Gordon model with output persistence, the correct delegated targets eliminate the constant average and state-contingent inflation biases, and a weight-liberal central bank removes the stabilization bias. For the new-Keynesian models, delegated targets, combined with the appropriate weight-liberal or –conservative central bank, can eliminate all three biases. The delegated targets may reflect backward- or forward-looking behavior, depending on the model.

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1. Introduction

The economic literature contains a strand that focuses on the inconsistency of optimal policy. Kydland and Prescott (1977) launch this whole literature by arguing that optimal policy proves inconsistent and showing that the inconsistency results from rational expectations. Afterwards, the literature proposes various solutions to the problem of the inconsistency. Using the Barro-Gordon (1983a,b) model, Rogoff (1985) suggests a conservative central banker, who puts more weight on inflation stabilization than output (employment) stabilization. Walsh (1995) proposes linear inflation contracts for central bankers to induce the socially optimal policy.

Later, the study of commitment and discretion in monetary policy moved beyond the static Barro-Gordon framework to models that introduce persistence in output and employment. Using an employment-persistence model, Svensson (1997) considers several different types of delegation (e.g., a state-contingent inflation target coupled with a weight-conservative central bank). The delegation scheme in Svensson (1997) induces the central bank to follow the optimal policy rule. Beetsma and Jensen (1999) argue that the state-contingent nature of the delegation scheme may undermine its credibility. They propose a nominal income growth target for the central bank, which although state-independent, nevertheless attains the optimal rule in Svensson's model. With general consensus on the forward-looking nature of macroeconomic models, Jensen (2002), Walsh (2003), Svensson and Woodford (2005) and Vestin (2006) among others, tackle the inconsistency issue in a new Keynesian model.

In each case, the proposed solution imposes two assumptions -- the central bank in practice operates with discretion and the central bank loss function must differ from the social one to mitigate or eliminate the inconsistency of optimal policy. This paper also imposes these two assumptions, and focuses on designing the central bank loss function. That is, the central bank loss function may differ in functional forms with different variables and parameters from those of the social loss function. For instance, the central bank loss function in Rogoff (1985) adopts the same functional form and target variables as those of the society, but different parameters, specifically more weight on inflation stabilization. Walsh (1995), Beetsma and Jensen (1999) and Svensson and Woodford (2005) modify the functional form by adding a term to the social loss function. According to the nature of the added term, the central bank loss function includes a linear inflation contract, constant nominal income growth targeting, and a "commitment to continuity and predictability." The central bank loss function in Vestin (2006), involving a price variable, adopts price-level targeting. Accordingly, the literature suggests that various central bank loss functions can replicate optimal policy by using discretionary policy.

This paper designs central bank loss functions by adopting the same functional form and target variables as those of the society, but with possibly different parameters (i.e., the targets for the inflation rate and the output gap, and the weight on output stabilization relative to inflation stabilization). First, the loss function explicitly incorporates the central bank's two target values and their relative importance. Second, the government delegates to the central bank short-term, state-contingent targets. The optimal solutions suggest two interesting parameters — the implied targets and the persistence parameter that governs the adjustment toward the implied targets. The implied targets exhibit a trade-off between targets and equal the long-run equilibrium values of target variables. The implied targets prove consistent with the models. Moreover, the implied targets generally differ from the social targets. Accordingly, we conjecture that the central bank's delegated targets, if consistent with the model and jointly achieved, should differ from the social ones. In addition, the implied targets emerge in the long run according to the persistence parameter. This suggests that the government delegates to the central bank short-term, state-contingent targets, which converge to the long-run implied targets at a rate controlled by the persistence parameter. Though the delegated targets are state-contingent, they only depend on the model's structural parameters and, thus, are feasible. Third, for the Barro-Gordon model with output persistence, the correct delegated targets eliminate the constant average and state-contingent inflation biases, and a weight-liberal central bank removes the stabilization bias.[1] For the new-Keynesian models, however, the delegated targets cannot eliminate the constant average and state-contingent inflation biases, until combined with the appropriate weight-liberal or –conservative central bank. At that point, all three biases disappear. In sum, based on the implied targets and the persistence parameter as well as the expectations nature of the Phillips curves, we design the target values and the weight of the central bank loss function to achieve optimal solutions. The next three sections will use three models to illustrate the approach.

The main results of the paper include the following. For the new-Keynesian models, the inflation target is forward-looking, if the Phillips curve is principally forward-looking, and vice

[1] Svensson (1997, p104) shows that discretionary policy exhibits three biases -- constant average and state-contingent inflation biases as well as stabilization bias. Once the three biases are removed, discretionary policy proves socially optimal.

versa.[2] Reversing the nature of the inflation target, optimal policy is forward-looking in a backward-looking model, and vice versa. This argument corresponds to Woodford (1999b), who demonstrates that optimal policy imparts inertia in a forward-looking model. In addition, the central bank weight may reflect conservatism or liberalism, depending on the structural parameters of the model as well as the persistence of the external shocks.

We organize the paper as follows. Section 2, 3, and 4 illustrate the approach to determining the delegated targets and the weight of the central bank loss function in a Barro-Gordon model with output persistence, a purely forward-looking new-Keynesian model, and a hybrid new-Keynesian model with forward- and backward-looking aspects. Section 5 summarizes.

2. Designing Central Bank Loss Function When Output Is Persistent

The Model and Its Socially Optimal Solution

The model follows Svensson (1997).[3] Society minimizes the following intertemporal loss function

(2.1) $$E_0\left(\sum_{t=1}^{\infty}\beta^{t-1}L_t\right),$$

where β ($0 < \beta < 1$) is the discount factor and E is the expectations operator. The period loss function equals the following

[2] A critical value exists that divides the Phillips curve into principally forward-looking or backward-looking specifications. See more details in Section 4.

[3] We take a pragmatic stand on the model, and simply adopt it without much description. Please read Svensson (1997) for more detail.

(2.2) $$L_t \equiv \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\left(x_t - x^*\right)^2\right],$$

where π is the inflation rate, x is the output gap, π^* is the socially desirable inflation rate, x^* is the socially desirable output gap, and λ is the social weight on output stabilization relative to inflation stabilization around their respective targets.

The economic structure includes an expectations-augmented Phillips curve with output persistence and rational expectations

(2.3) $$x_t = \eta x_{t-1} + \alpha\left(\pi_t - \pi_t^e\right) + u_t, \text{ and}$$

(2.4) $$\pi_t^e = E_{t-1}\pi_t,$$

where η ($0 \leq \eta < 1$) measures the degree of output gap persistence, α is the response of output gap to unexpected inflation, π_t^e denotes inflation expectations in period t–l of the inflation rate in period t, and u_t is an independently and identically distributed supply shock with mean 0 and variance σ^2.

The central bank minimizes the social intertemporal loss function (2.1) with period loss function (2.2) subject to equations (2.3) and (2.4). The socially optimal solution under commitment equals[4]

(2.5a) $$\pi_t = \pi^* + d^* u_t, \text{ and}$$

(2.5b) $$x_t = \eta x_{t-1} + (1 + \alpha d^*) u_t,$$

where $d^* = -\dfrac{\lambda\alpha}{1 - \beta\eta^2 + \lambda\alpha^2}$.

The optimal solutions in equations (2.5a) and (2.5b) consist of two parts – the systematic

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[4] See Svensson (1997).

evolution paths of the inflation rate and the output gap, $\left\{\pi^*, \eta x_{t-1}\right\}_{t=1}^{\infty}$, and reactions to supply shocks. The systematic evolution paths converge to $\left(\pi^*, 0\right)$. So we interpret π^* and 0 as long-run implied targets of the inflation rate and the output gap, which we specify as π^a and x^a, respectively. Moreover, the rates of convergence differ. That is, the inflation rate hits its implied target, π^*, immediately with zero persistence, and the output gap converges to its implied target, 0, with persistence η. To understand zero implied output gap target, which differs from the social target, x^*, consider the Phillips curve (2.3) and the rational expectations assumption in equation (2.4). The equilibrium output gap requires that

(2.6) $$\bar{x} = \eta \bar{x} \text{, i.e. } \bar{x} = x^a = 0,$$

where $\bar{x}$ denotes the equilibrium output gap. That is, the implied target (x^a) in the optimal solution equals the equilibrium value, and proves harmonious with the Phillips curve.

As is well known, the optimal solution is time-inconsistent. Implementing the optimal policy requires either commitment or mechanism design. A central bank loss function that differs from the social loss function may serve as a mechanism to reduce or even eliminate the inconsistency. Accordingly, the next sub-section designs the central bank loss function.

Designing the Central Bank Loss Function

We adopt a central bank loss function, which adopts the same functional form and target variables as those of the society but with possibly different inflation rate target π_t^b, output gap target x_t^b, and weight λ^b. We permit state-contingent targets for the inflation rate and the output gap, but not for the weight. That is, the central bank period loss function equals

(2.7) $$L_t^b = \frac{1}{2}\left[\left(\pi_t - \pi_t^b\right)^2 + \lambda^b\left(x_t - x_t^b\right)^2\right].$$

The central bank minimizes the intertemporal loss function (2.1), but with period loss function (2.7). That is, we assume both society and the central bank weight the future with same importance, β.

The optimal solutions suggest that the realized values of target variables in the long-run equal the implied targets, π^* and 0, that differ from the social targets, π^* and x^*. That is, the best outcomes that we can achieve equal the implied targets, which are consistent with the model. Moreover, they only emerge in the long run subject to the persistence parameter. So we conjecture that the appropriate targets for the central bank are short-term and state-contingent, which converge to the implied targets. As a result, we set the systematic evolution paths in the optimal solution equal to the central bank's targets

(2.8a) $$\pi_t^b = \pi^* \text{ and}$$

(2.8b) $$x_t^b = \eta x_{t-1}.$$

Though the central bank's delegated targets are state-contingent, they only depend on the model's structural parameters and are, thus, feasible. Intuitively, with short-term natural output gap ηx_{t-1}, not the social target x^*, as the output gap target, the central bank does not possess an incentive to produce surprise inflation $\left(\pi_t - \pi_t^e\right)$ to raise the output gap above the short-term natural gap ηx_{t-1}. As a result, the central bank can realize the targets π^* and ηx_{t-1}, on average. The formal calculation of discretionary policy with target values in equations (2.8a) and (2.8b) verifies the intuition. The discretionary solution equals [5]

(2.9a) $$\pi_t = \pi^* - \frac{\alpha\lambda^b}{1+\alpha^2\lambda^b} u_t \text{, and}$$

(2.9b) $$x_t = \eta x_{t-1} + \frac{1}{1+\alpha^2\lambda^b} u_t .$$

The discretionary policy in equations (2.9a) and (2.9b) eliminates the constant average and state-contingent inflation biases (see Svensson, 1997, 104). The two targets in equations (2.8a) and (2.8b) prove meaningful for monetary policy. That is, the central bank, though operating with discretion, remains accountable for monetary policy because it can reach the delegated targets, on average.

As we noted, the discretionary policy in equations (2.9a) and (2.9b) eliminates constant average and state-contingent inflation biases. We also need to remove the stabilization bias (i.e., it responds to supply shocks optimally). Equating discretionary policy in equations (2.9a) and (2.9b) with the optimal solutions in equations (2.5a) and (2.5b) produces

(2.10) $$\lambda^b = \frac{\lambda}{1-\beta\eta^2} .$$

In sum, with appropriate state-contingent targets, discretionary policy eliminates the constant average and state-contingent inflation biases, and a weight-liberal central bank also removes the stabilization bias. As a result, discretionary policy proves socially optimal.

The central bank's weight λ^b on output stabilization relative to inflation stabilization depends on the structural parameters of the social loss function and the Phillips curve (i.e., the

[5] See equation (A.11) and its derivation in Appendix A.

discount factor β, the social weight λ, and the output gap persistence η).[6] Furthermore, we report the following conditions

(2.11a) $\partial\lambda^b/\partial\beta > 0$ if $\eta \neq 0$,

(2.11b) $\partial\lambda^b/\partial\eta > 0$ if $\beta \neq 0$,

(2.11c) $\lambda^b > \lambda$ and $\partial\lambda^b/\partial\lambda > 0$ if $\beta \neq 0$ and $\eta \neq 0$, and

(2.11d) $\lambda^b = \lambda$ if $\beta = 0$ and/or $\eta = 0$.

Consider condition (2.11a). Given output persistence, a more important future implies that optimal policy places more weight on output stabilization. Intuitively, the loss caused by the output gap will persist into future, if output exhibits persistence. Therefore, a more important future implies more weight must be placed on output stabilization to reduce the future losses caused by output persistence.

Consider condition (2.11b). As long as society places some weight on the future, a more persistent output implies that optimal policy places more weight on output stabilization. Intuitively, a non-zero output gap will cause current and future losses, if output exhibits persistence. And, a more persistent the output gap implies that more future losses will occur. Accordingly, more weight on output stabilization reduces future losses.

Consider condition (2.11c). Contrary to the usual recommendation of appointing a conservative central banker, the weight in equation (2.10) suggests a liberal central banker, who places more weight on output stabilization than society, as long as society cares for future and

[6] Here, we assume that supply shocks equal white noise. Thus, the weight does not reflect the characteristic of the supply shocks. We discuss the persistence of shocks in next two sections.

output exhibits persistence. The intuition conforms to that for conditions (2.11a) and (2.11b) above. In addition, the more weight the society places on output stabilization, the more weight the central bank must place on output stabilization. The intuition is straightforward.

Consider condition (2.11d). If society does not care about the future and/or if output exhibits no persistence, then the central bank places the same weight on output stabilization as society, since the central bank does not need to balance current and future losses. The central bank does not exhibit conservatism or liberalism.

3. Designing Central Bank Loss Function in a Purely Forward-Looking, New-Keynesian Model

Researchers developed and applied new-Keynesian models in the past decade. This section uses a purely forward-looking new-Keynesian model to illustrate the approach to designing central bank loss functions.[7]

The Model and Its Socially Optimal Solution

The social intertemporal loss function equals[8]

(3.1) $$\mathbb{L} = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t\right)$$

with the same period loss function as in Section 2 [i.e., equation (2.2)]

(3.2) $$L_t = \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\left(x_t - x^*\right)^2\right].$$

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[7] We do not take a stand on the validity of the models in this section and next. We simply adopt them without much discussion. See Clarida *et al.* (1999) for more details.

[8] Equation (3.1) slightly differs from equation (2.1) in that in Section 2, we form the rational expectation in the previous period and in this section at the beginning of the present period. We just follow usual definitions in literature for convenience.

We model aggregate supply as an expectations-augmented Phillips curve with purely forward-looking expectations[9]

(3.3) $$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t ,$$

where π, x, and β are defined as in previous section, κ ($\kappa > 0$) is the sensitivity of the inflation rate to the output gap, u_t is a cost-push shock following AR(1) process

(3.4) $$u_t = \rho u_{t-1} + \hat{u}_t ,$$

where $0 \le \rho < 1$, and $\hat{u}_t$ is a white noise residual.

We do not introduce aggregate demand (IS curve), which contains a nominal interest rate, the policy instrument. Once we determine the optimal paths of $\{\pi_t, x_t\}_{t=0}^{\infty}$ according to the social loss function and the Phillips curve, both of which do not respond to the interest rate, then we can pin down the optimal path of interest rates through the IS curve. So what is critical for policy is the Phillips curve.

The consolidated first-order condition of optimal policy under the social intertemporal loss function (3.1) with period loss function (3.2) subject to the Phillips curve (3.3) equals[10]

(3.5a) $$x_0 - x^* = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right) \qquad \text{for } t = 0 \text{, and}$$

(3.5b) $$E_0 x_t - E_0 x_{t-1} = -\frac{\kappa}{\lambda}\left(E_0 \pi_t - \pi^*\right) \qquad \text{for } t \ge 1.$$

Combining the first-order conditions (3.5a) and (3.5b) and the Phillips curve (3.3) leads to the

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[9] The next section considers both forward- and backward-looking expectations in the Phillips curve. Equation (3.3) also differs slightly from equation (2.3) in that in Section 2, we form rational expectations in the previous period and in this section at the beginning of the present period.

[10] See equations (B.3a) and (B.3b) in Appendix B.

socially optimal solution[11]

(3.6a) $$\left(\pi_0 - \pi^*\right) = \frac{\lambda}{\kappa}(1-\delta)\left(x^* - \bar{x}\right) + \frac{\delta}{(1-\delta\beta\rho)}u_0, \text{ for } t = 0,$$

(3.6b) $$\left(x_0 - x^a\right) = \delta\left(x^* - x^a\right) - \frac{\kappa\delta}{\lambda(1-\delta\beta\rho)}u_0, \text{ for } t = 0,$$

(3.7a) $$\left(\pi_t - \pi^*\right) = \delta\left(\pi_{t-1} - \pi^*\right) - \frac{\delta(1-\rho)}{\rho(1-\delta\beta\rho)}u_t, \text{ for } t \geq 1, \text{ and}$$

(3.7b) $$\left(x_t - x^a\right) = \delta\left(x_{t-1} - x^a\right) - \frac{\kappa\delta}{\lambda(1-\delta\beta\rho)}u_t, \text{ for } t \geq 1,$$

where δ (0<δ<1) is the smaller root of the characteristic equation

(3.8a) $$\beta\delta^2 - b\delta + 1 = 0,$$

(3.8b) $$b \equiv 1 + \beta + \frac{\kappa^2}{\lambda} \text{ and}$$

(3.8c) $$x^a \equiv \frac{1-\beta}{\kappa}\pi^*.$$

The parameter x^a defines the implied output gap target. See Appendix B, equation (B.17).

Equations (3.6a), (3.6b), (3.7a), and (3.7b) suggest that the socially optimal solution is time inconsistent. We adopt optimality from the timeless perspective as the benchmark.[12] That is, we design the central bank loss function to replicate the solutions in equations (3.7a) and (3.7b) for all periods. To do this, we first analyze the properties of these solutions.

The solutions in equations (3.7a) and (3.7b) consist of two parts – the systematic evolution paths of the inflation rate and the output gap, $\left\{\pi^* + \delta\left(\pi_{t-1} - \pi^*\right), x^a + \delta\left(x_{t-1} - x^a\right)\right\}_{t=1}^{\infty}$,

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[11] See equations (B.15a), (B.15b), (B.24a), and (B.24b) and their derivation in Appendix B.

[12] Woodford (1999a) introduces the concept of optimality from a "timeless perspective," which means the policy the central bank "would have wished to commit itself to at a date far in the past."

and reactions to cost-push shocks. The systematic evolution paths converge to $\left(\pi^{*}, x^{a}\right)$ with the same persistence parameter δ. So we interpret π^{*} and x^{a} as long-run implied targets of the inflation rate and the output gap. To understand the implied targets, consider the Phillips curve (3.3). The equilibrium relationship of the inflation rate and the output gap must conform to

(3.9) $$\overline{\pi} = \kappa \overline{x} + \beta \overline{\pi} ,$$

where $\overline{\pi}$ and $\overline{x}$ denote the long-run equilibrium values of the inflation rate and the output gap. Equation (3.9) is just equation (3.8c), where $\overline{\pi} = \pi^{a} = \pi^{*}$ and $\overline{x} = x^{a}$. Therefore, on the one hand, a trade-off exists between the implied targets; on the other hand, we can interpret the implied targets as the long-run equilibrium values of the target variables. Intuitively, the Phillips curve imposes a trade-off between the inflation rate and the output gap. Thus, the implied targets for the inflation rate and the output gap also must exhibit this same trade-off. As such, the implied targets are consistent with the model.

In addition, the implied output gap target generally differs from the social one, and we observe that they coincide when the social targets satisfy the trade-off.[13] Specifically,

$$\pi^{*} = \kappa x^{*} + \beta \pi^{*}, \text{ i.e. } x^{*} = \frac{1-\beta}{\kappa}\pi^{*} .$$

Then, by equation (3.8c) we know that

$$x^{a} \equiv \frac{1-\beta}{\kappa}\pi^{*} = x^{*} .$$

In this respect, the social targets are generally inconsistent with the model.

The persistence parameter δ that governs the dynamic adjustment of the inflation rate and

[13] For Barro-Gordon Model with output persistence, the Phillips curve in equilibrium simplifies to $\overline{x} = 0$. If the social targets satisfy the Phillips curve in equilibrium, i.e. $x^{*} = 0$, then obviously the social targets coincide with the implied targets. That is, $\pi^{*} = \pi^{a}$ and $x^{*} = 0 = x^{a}$.

the output gap to their respective equilibria depends on the structural parameters (i.e., the discount factor β, the weight λ on output stabilization, and the sensitivity κ of the inflation rate to the output gap) of the social loss function and the Phillips curve, but does not depend on the cost-push shock. Furthermore, we report the following conditions

(3.10a) $\partial\delta/\partial\beta < 0$,

(3.10b) $\partial\delta/\partial\lambda > 0$, and

(3.10c) $\partial\delta/\partial\kappa < 0$.[14]

Consider condition (3.10a). A more important future implies a less persistent evolution of the inflation rate and the output gap. Intuitively, if the future is more important, the inflation rate and the output gap must evolve more quickly to their targets to reduce future losses. That is, the inflation rate and the output gap evolve with less persistence.

Consider condition (3.10b). A more important role for output stabilization implies a more persistent evolution of the inflation rate and the output gap. Intuitively, when output stabilization is more important, a smaller change of the output gap occurs as the consolidated first-order conditions suggest.[15] As a result, the output gap and, thus, the inflation rate evolve more sluggishly.

Consider condition (3.10c). A more sensitive inflation rate response to the output gap implies a less persistent evolution of the inflation rate and the output gap. Intuitively, with the inflation rate more sensitive to the output gap, the inflation rate and the output gap will evolve

[14] See equations (C.8), (C.11), and (C.13) and their derivations in Appendix C.

[15] See equations (B.3a) and (B.3b) in Appendix B.

more quickly to their targets. That is, the inflation rate and the output gap evolve with less persistence.

Designing the Central Bank Loss Function

Optimal solutions in equations (3.7a) and (3.7b) suggest that the implied targets are the best long-run outcomes, and they emerge in the long run according to the persistence parameter. Thus, we delegate state-contingent targets to the central bank. That is, we set the systematic evolution paths in the optimal solutions in equations (3.7a) and (3.7b) as the targets of the central banker to direct target variables to evolve along the optimal paths. That is,

(3.11a) $\pi_t^b = \pi^* + \delta\left(\pi_{t-1} - \pi^*\right)$ and

(3.11b) $x_t^b = x^a + \delta\left(x_{t-1} - x^a\right)$.

Though the delegated targets are state-contingent, they only depend on the model's structural parameters and, thus, are feasible. The consolidated first-order condition of discretionary policy under the loss function (2.7) with targets in equations (3.11a) and (3.11b) subject to the Phillips curve (3.3) equals[16]

(3.12)
$$\begin{aligned}&\kappa\left\{\left[\left(\pi_t - \pi^*\right) - \delta\left(\pi_{t-1} - \pi^*\right)\right] - \beta\delta\left[\left(E_t\pi_{t+1} - \pi^*\right) - \delta\left(\pi_t - \pi^*\right)\right]\right\}\\ &\quad + \lambda^b\left\{\left[\left(x_t - x^a\right) - \delta\left(x_{t-1} - x^a\right)\right] - \beta\delta\left[\left(E_t x_{t+1} - x^a\right) - \delta\left(x_t - x^a\right)\right]\right\} = 0\end{aligned}.$$

Combining the first-order condition (3.12) and the Phillips curve (3.3) generates discretionary policy, taking the following form:

(3.13a) $\left(\pi_t - \pi^d\right) = \delta^d\left(\pi_{t-1} - \pi^d\right) + gu_t$, and

[16] See equation (D.4) in Appendix D. We change the subscripts, which equal period 0 in Appendix D, to period t here. Discretion policy means that the central bank re-makes policy each period.

(3.13b) $$\left(x_t - x^d\right) = \delta^d \left(x_{t-1} - x^d\right) + hu_t ,$$

where π^d and x^d denote implied targets of discretionary policy, δ^d the evolution persistence, g and h reactions of the inflation rate and the output gap to cost-push shocks. Similarly, δ^d is a less-than-one root of the characteristic equation of the first-order condition (3.12) and the Phillips curve (3.3).[17] The discretionary solutions in equations (3.13a) and (3.13b) exhibit an constant average inflation bias, $\left(1-\delta^d\right)\pi^d - \pi^*$, and an state-contingent inflation bias, $\delta^d \pi_{t-1}$. A stabilization bias also exists, in that generally $g \neq -\delta(1-\rho)/\left[\rho(1-\delta\beta\rho)\right]$.

If $\pi^d = \pi^*$ and $\delta^d = \delta$, then the discretionary policy eliminates the constant average and state-contingent inflation biases. But, obviously, δ^d depends on λ^b, because the characteristic equation of the discretionary policy involves λ^b.[18] That is, correct delegated targets to the central bank are not enough to eliminate the constant average and state-contingent inflation biases, and the weight is also related.[19] In addition, solving for the discretionary policy proves too complicated, using the method in Section 2. Consequently, we adopt a different method. Discretionary policy should replicate the optimal policy. If the optimal solutions in equations (3.7a) and (3.7b) satisfy the first-order condition of discretionary policy, equation (3.12), then the replication occurs. Using equations (3.7a), (3.7b) and $E_t u_{t+1} = \rho u_t$ in equation (3.12) generates[20]

(3.14) $$\lambda^b = -\frac{(1-\rho)}{\rho}\lambda .$$

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[17] We do not discuss whether a less-than-one root exists. We just assume that it exists.

[18] For the optimal solutions, the persistence parameter, δ, depends on λ. See equations (3.8a) and (3.8b).

[19] For Barro-Gordon model with output persistence, with the correct delegated targets, equations (2.8a) and (2.8b), discretionary policy, equation (2.9a), eliminates the two biases.

[20] See equation (D.6) and its derivation in Appendix D.

The weight λ^b is negative and, thus, infeasible.

The problem with applying our intuitive approach in Section 2, where the model is backward-looking, to the current model involves the forward-looking nature of the Phillips curve (3.3). Thus, an alternative intuitive solution may require that the inflation target of the central bank may involve a term $E_t\pi_{t+1}$ in order to reflect this forward-looking nature. At the same time, we require that the inflation rate evolve with an optimal persistence δ. As a result, we try the following targets

(3.15a) $\pi_t^b = \pi^* + \left(E_t\pi_{t+1} - \pi^*\right)/\delta$ and

(3.15b) $x_t^b = x^a + \delta\left(x_{t-1} - x^a\right)$.

The consolidated first-order condition of discretionary policy under the loss function (2.7) with targets in equations (3.15a) and (3.15b) subject to the Phillips curve (3.3) equals[21]

$$\kappa\left[\left(\pi_t - \pi^*\right) - \left(E_t\pi_{t+1} - \pi^*\right)/\delta\right] + \lambda^b\left\{\left[\left(x_t - x^a\right) - \delta\left(x_{t-1} - x^a\right)\right] - \beta\delta\left[\left(E_t x_{t+1} - x^a\right) - \delta\left(x_t - x^a\right)\right]\right\} = 0 \quad (3.16)$$

Combining the first-order condition (3.16) and the Phillips curve (3.3) generates discretionary policy under the delegated targets in equations (3.15a) and (3.15b). The discretionary solutions also have the same forms to those of equations (3.13a) and (3.13b), and exhibit constant average and state-contingent inflation biases as well as a stabilization bias. A necessary condition that the discretionary policy replicates the optimal policy is that the optimal solutions in equations (3.7a) and (3.7b) satisfy the first-order condition (3.16). Using equations (3.7a), (3.7b) and

[21] See equation (E.3) in Appendix E. We change the subscripts, which equal period 0 in Appendix E, to period t here.

$E_t u_{t+1} = \rho u_t$ in equation (3.16) produces[22]

(3.17) $$\lambda^b = \frac{1-\rho}{\delta(1-\delta\beta\rho)}\lambda,$$

where λ^b is positive and feasible. In sum, under the loss function (2.7) with delegated targets in equations (3.15a) and (3.15b) and the weight in equation (3.17), one of the discretionary policies is socially optimal.

Consider the weight in equation (3.17). In addition to the structural parameters (i.e., the discount factor β, the social weight λ, and the sensitivity κ), the central bank weight λ^b also depends on the cost-push shock persistence ρ. Furthermore, we report the following conditions[23]

(3.18a) $\partial\lambda^b/\partial\rho < 0$,

(3.18b) $\partial\lambda^b/\partial\beta > 0$,

(3.18c) λ^b does not respond monotonically to λ, and

(3.18d) λ^b does not respond monotonically to κ.

Consider condition (3.18a). A more persistent cost-push shock implies that the policy maker must place less weight on output stabilization. To see this, iterate the Phillips curve (3.3) forward as follows

(3.19) $$\pi_t = E_t \sum_{i=0}^{\infty} \beta^i \left(\kappa x_{t+i} + u_{t+i}\right).$$

Inflation depends entirely on current and expected future output gaps and cost-push shocks, because of the purely forward-looking nature of the Phillips curve. Thus, more persistent

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[22] See equation (E.5) in Appendix E.

[23] See equations (F.2), (F.7), (F.8), and (F.9) in Appendix F.

cost-push shocks imply that more losses occur because more inflation increases emerge, driven by current and future cost-push shocks. To reduce the losses caused by current and future inflation, the policy makers must place more weight on inflation stabilization and, thus, less weight on output stabilization.

Consider condition (3.18b). A more important future implies that the policy makers must place more weight on output stabilization. Referring to equation (3.19), inflation depends entirely on current and expected future output gaps and cost-push shocks. When the future is more important, future output gaps exert more influence on inflation. Then, more weight on output stabilization will stabilize more current and future output gaps and, thus, stabilize inflation.

Consider condition (3.18c). The central bank weight λ^b does not respond monotonically to the social weight λ. That is, λ^b can equal, fall below, or exceed λ. We define a critical value of $\rho_{\lambda c}$ for which λ equals λ^b. Specifically,

(3.20)
$$\left\{\begin{array}{ll} \lambda^b < \lambda & \text{if } \rho > \rho_{\lambda c} \\ \lambda^b = \lambda & \text{if } \rho = \rho_{\lambda c} \\ \lambda^b > \lambda & \text{if } \rho < \rho_{\lambda c} \end{array}\right\}, \text{ where } \rho_{\lambda c} \equiv \frac{1-\delta}{1-\beta\delta^2}.$$

First, the critical value $\rho_{\lambda c}$ ($0 < \rho_{\lambda c} < 1$) depends on structural parameters of the social loss function and the Phillips curve. Second, whether the central bank exhibits conservatism or liberalism depends on social loss function, the Phillips curve, and even the cost-push shock persistence. This result counters the usual claim that the central bank must prove more conservative than society. Actually, the result that $\partial\lambda^b/\partial\rho < 0$ implies equation (3.20) to some extent. That is, higher persistence of the cost-push shock results in a lower central bank weight on output stabilization and, thus, the central bank weight may fall below that of society.

Consider condition (3.18d). The sign of $\partial\lambda^b/\partial\kappa$ depends on $(1-2\delta\beta\rho)$. Define the critical value for ρ as follows:

$$(3.21) \qquad \rho_{\kappa c} \equiv \frac{1}{2\beta\delta} = \frac{1}{b-\sqrt{b^2-4\beta}}.$$

This critical value depends on the structural parameters of the social loss function and the Phillips curve, but may exceed 1. That is,[24]

$$(3.22) \qquad \rho_{\kappa c}\begin{cases} \geq 1, & \text{if } \beta \leq \dfrac{1}{2}+\dfrac{\kappa^2}{\lambda} \\ <1, & \text{if } \beta > \dfrac{1}{2}+\dfrac{\kappa^2}{\lambda} \end{cases}.$$

Accordingly, three situations exist, two imply that $\partial\lambda^b/\partial\kappa \geq 0$, and one that $\partial\lambda^b/\partial\kappa < 0$. More specifically,

$$(3.23) \qquad \frac{\partial\lambda^b}{\partial\kappa}\begin{cases} \geq 0, & \begin{cases} \text{if } \beta \leq \dfrac{1}{2}+\dfrac{\kappa^2}{\lambda}, \text{ then } \rho_{\kappa c} \geq 1, \text{ then } \rho \leq \rho_{\kappa c} \\ \text{if } \beta > \dfrac{1}{2}+\dfrac{\kappa^2}{\lambda} \text{ and } \rho \leq \rho_{\kappa c} \end{cases} \\ <0, & \text{if } \beta > \dfrac{1}{2}+\dfrac{\kappa^2}{\lambda} \text{ and } \rho > \rho_{\kappa c} \end{cases}.$$

Consider the two situations where $\partial\lambda^b/\partial\kappa \geq 0$. First, society does not care so much about the future (i.e., $\beta \leq 1/2+\kappa^2/\lambda$), no matter how persistent the cost-push shock. Second,

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[24] If $\beta \leq \frac{1}{2}+\frac{\kappa^2}{\lambda}$, then $\left(\beta+\frac{\kappa^2}{\lambda}\right)^2 \leq b^2-4\beta$. Then $\left(\beta+\frac{\kappa^2}{\lambda}\right)-\sqrt{b^2-4\beta} \leq 0$. Then $\left(1+\beta+\frac{\kappa^2}{\lambda}\right)-\sqrt{b^2-4\beta} \leq 1$, i.e., $b-\sqrt{b^2-4\beta} \leq 1$. Thus, $\rho_{\kappa c} = \frac{1}{b-\sqrt{b^2-4\beta}} \geq 1$.

society cares more about the future (i.e., $\beta > 1/2 + \kappa^2/\lambda$), and the cost-push shock exhibits lower persistence (i.e., $\rho < \rho_{\kappa c}$). In both situations, a more sensitive response of the inflation rate to the output gap implies that policy makers must place more weight on output stabilization. To see this, consider the Phillips curve, $\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t$. Under the two situations, the central bank can ignore the terms, $\beta E_t \pi_{t+1}$ and u_t, to some extent, and exploit the sensitivity parameter κ of how the inflation rate responds to the output gap.

Consider the situation where $\partial \lambda^b / \partial \kappa < 0$. Once again, society cares about the future (i.e., $\beta > 1/2 + \kappa^2/\lambda$), and the cost-push shock exhibits higher persistence, (i.e., $\rho > \rho_{\kappa c}$). In this situation, a more sensitive response of the inflation rate to the output gap implies that the policy makers must place less weight on output stabilization. Intuitively, in this situation, the central bank can no longer exploit the sensitivity of the inflation rate to the output gap, because the terms $\beta E_t \pi_{t+1}$ and u_t exert much more influence directly on current and future inflation. Accordingly, the policymakers must place more weight directly on inflation stabilization to counteract the sensitivity of the inflation rate to the output gap. A more sensitive response of the inflation rate to the output gap implies that the policy makers must place more weight on inflation stabilization and, thus, less weight on output stabilization.

4. Designing Central Bank Loss Function in a Hybrid New-Keynesian Model

This section combines the possibility of backward- and forward-looking inflation expectations into a hybrid model. This model reduces to entirely forward-looking or entirely backward-looking models by choosing the extreme values of the parameter that indexes expectations across the backward- and forward-looking dimensions.

The Model and Its Socially Optimal Solution

Consider the following generalization of the Phillips curve

(4.1) $$\pi_t = \kappa x_t + \phi \pi_{t-1} + (1-\phi)\beta E_t \pi_{t+1} + u_t,$$

where parameter ϕ indexes the degree of lagged versus expected future inflation rates. The socially optimal solution for $t \geq 1$ equals [25]

(4.2a) $$\left(\pi_t - \pi^a\right) = \delta\left(\pi_{t-1} - \pi^a\right) + \left[\phi\left(1-\beta\rho^2\right) - (1-\rho)\right]\frac{\delta}{(\delta\rho d - a)}u_t, \text{ and}$$

(4.2b) $$\left(x_t - x^a\right) = \delta\left(x_{t-1} - x^a\right) - \frac{\kappa\delta\rho}{\lambda(\delta\rho d - a)}u_t,$$

where δ is a root of the characteristic equation[26]

(4.3a) $$a\beta^2\delta^4 - \beta\delta^3 + b\delta^2 - \delta + a = 0 \quad \text{with}$$

(4.3b) $$a \equiv \phi(1-\phi),$$

(4.3c) $$b \equiv 1 + \frac{\kappa^2}{\lambda} + \phi^2\beta + (1-\phi)^2\beta = 1 + \beta + \frac{\kappa^2}{\lambda} - 2a\beta,$$

(4.3d) $$d \equiv a\beta^2\left(\delta^2 + \delta\rho + \rho^2\right) - \beta(\delta + \rho) + b,$$

(4.3e) $$\pi^a \equiv \frac{\kappa\left[\kappa\pi^* + \lambda\phi(1-\beta)x^*\right]}{\kappa^2 + \lambda a(1-\beta)^2}, \text{ and}$$

(4.3f) $$x^a \equiv \frac{(1-\phi)(1-\beta)}{\kappa}\pi^a.$$

Obviously, when ϕ = 0, the characteristic equation (4.3a), the implied output target in equation

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[25] See equations (G.28) and their derivations in Appendix G.

[26] We do not discuss multiple equilibria or consider whether a convergent equilibrium exists. Rather, we just assume that a root exists between 0 and 1.

(4.3f) and the optimal solutions in equations (4.2a) and (4.2b) reduce to equations (3.8a), (3.8c), (3.7a), and (3.7b), since $a = 0$.[27]

Once again, we adopt the optimality from a timeless perspective as the benchmark. That is, we design the central bank loss function to replicate the solutions in equations (4.2a) and (4.2b) for all periods. To do this, we first analyze the properties of these solutions.

Similar to Section 3, solutions in equations (4.2a) and (4.2b) consist of two parts – the systematic evolution paths of the inflation rate and the output gap, and reactions to cost-push shocks. The systematic evolution paths converge to $\left(\pi^a, x^a\right)$ with the same persistence parameter δ. So we interpret π^a and x^a as the long-run implied targets of the inflation rate and the output gap. To understand the implied targets, consider the Phillips curve (4.1). The equilibrium relationship of the inflation rate and the output gap must conform to

$$\bar{\pi} = \kappa\bar{x} + \phi\bar{\pi} + \left(1-\phi\right)\beta\bar{\pi}, \tag{4.4}$$

where $\bar{\pi}$ and $\bar{x}$ denote the equilibrium values of the inflation rate and the output gap. Equation (4.4) is just equation (4.3f), where $\bar{\pi} = \pi^a$ and $\bar{x} = x^a$. Similarly, a trade-off exists between the implied targets, and we can interpret the implied targets as the long-run equilibrium values. The intuition matches that in Section 3. The implied targets prove consistent with the model.

Moreover, the implied targets generally differ from the social ones, and we observe that they coincide when the social targets satisfy the trade-off. Specifically, assume

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[27] Here, we show that the solutions in equations (4.2a) and (4.2b) reduce to equations (3.7a) and (3.7b). If $\phi = 0$, then $a = 0$, $d = b - \beta\left(\delta + \rho\right)$, and $d\delta = b\delta - \beta\delta^2 - \delta\beta\rho$. Using characteristic equation (3.8a), $b\delta - \beta\delta^2 = 1$. Therefore, $d\delta = 1 - \delta\beta\rho$ or $1/d = \delta/\left(1-\delta\beta\rho\right)$. As a result, equations (4.2a) and (4.2b) reduce to equations (3.7a) and (3.7b).

$$\pi^* = \kappa x^* + \phi\pi^* + (1-\phi)\beta\pi^*, \text{ i.e. } x^* = \frac{(1-\phi)(1-\beta)}{\kappa}\pi^*.$$

Then,

$$\pi^a \equiv \frac{\kappa\left[\kappa\pi^* + \lambda\phi(1-\beta)x^*\right]}{\kappa^2 + \lambda a(1-\beta)^2} = \frac{\kappa}{\kappa^2 + \lambda a(1-\beta)^2}\left[\kappa\pi^* + \lambda\phi(1-\beta)\frac{(1-\phi)(1-\beta)}{\kappa}\pi^*\right] = \pi^*, \text{ and}$$

$$x^a \equiv \frac{(1-\phi)(1-\beta)}{\kappa}\pi^a = \frac{(1-\phi)(1-\beta)}{\kappa}\pi^* = x^*.$$

In this respect, the social targets are generally inconsistent with the model.

Equations (4.3a), (4.3b), and (4.3c) determine δ.[28] No matter whether a root $\delta \in (0,1)$ exists for the characteristic equation (4.3a), δ only depends on the structural parameters (i.e., the discount factor β, the weight λ on output stabilization, the sensitivity κ of inflation to output gap, and the lagged degree ϕ of inflation) of the social loss function and the Phillips curve, and does not depend on the cost-push shock.

Consider the two polar models—a purely forward-looking model ($\phi = 0$) and a purely backward-looking model ($\phi = 1$). The characteristic equations reduce to the same equation (3.7a) in the two polar models, since $a = 0$. That is, the inflation rate and the output gap evolve with the same persistence along the optimal paths in the two polar models, although toward different implied targets. Furthermore, the output gaps respond in the same way and the inflation rates respond differently to the cost-push shocks in the two polar models. Specifically,

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[28] We cannot determine whether a root $\delta \in (0,1)$ exists in the characteristic equation (4.3a) for any lagged degree $\phi \in (0,1)$. But for the two polar cases of $\phi = 0$ and $\phi = 1$, the characteristic equation (4.3a) reduces to (3.8a), and a root $\delta \in (0,1)$ does exist.

(4.5a) $$\phi = 0 \begin{cases} \left(\pi_t - \pi^*\right) = \delta\left(\pi_{t-1} - \pi^*\right) - \dfrac{1-\rho}{\rho d} u_t \\ \left(x_t - x^a\right) = \delta\left(x_{t-1} - x^a\right) - \dfrac{\kappa}{\lambda d} u_t \end{cases}$$, where δ satisfies (3.7a) , and

(4.5b) $$\phi = 1 \begin{cases} \left(\pi_t - \pi^a\right) = \delta\left(\pi_{t-1} - \pi^a\right) + \dfrac{1-\beta\rho}{d} u_t \\ \left(x_t - 0\right) = \delta\left(x_{t-1} - 0\right) - \dfrac{\kappa}{\lambda d} u_t \end{cases}$$, where δ satisfies (3.7a) .

In equation (4.5a), $x^a = (1-\beta)\pi^* / \kappa$, while in equation (4.5b), $\pi^a = [\kappa\pi^* + \lambda(1-\beta)x^*] / \kappa$.

To understand the different responses of the inflation rates to the cost-push shocks in the two polar models, we transform the inflation rate expressions as follows:

(4.6a) $$\phi = 0 : \left(E_0\pi_t - \pi^*\right) = \delta\left(E_0\pi_{t-1} - \pi^*\right) + \frac{1}{d}\left(E_0 u_t - E_0 u_{t-1}\right)$$, and

(4.6b) $$\phi = 1 : \left(E_0\pi_t - 0\right) = \delta\left(E_0\pi_{t-1} - 0\right) + \frac{1}{d}\left(E_0 u_t - \beta E_0 u_{t+1}\right).$$

That is, the inflation rates respond to the change of the cost-push shocks at current and previous periods in a purely forward-looking model (ϕ=0), and to the difference between current shock and discounted shock of next period in a purely backward-looking model (ϕ=1). Woodford (1999b) demonstrates that an optimal policy imparts inertia when expectations are forward-looking. We argue that an optimal policy imparts forward-looking nature in a backward-looking model. The consolidated first-order conditions also reveal this point.[29]

(4.7a) $$\phi = 0 : E_0 x_t - E_0 x_{t-1} = -\frac{\kappa}{\lambda}\left(E_0\pi_t - \pi^*\right)$$, and

(4.7b) $$\phi = 1 : E_0 x_t - \beta E_0 x_{t+1} = -\frac{\kappa}{\lambda}\left(E_0\pi_t - \pi^*\right) + \left(1-\beta\right)x^* .$$

Intuitively, when the private sector looks backward, the central bank must lead the private sector to the optimal path. Accordingly, optimal policy looks forward in a backward-looking model.

[29] See equation (G.3b) and its derivation in Appendix G.

In sum, optimal policy looks forward (backward) in a backward-(forward-) looking model.

Designing the Central Bank Loss Function

The Phillips curve involves backward and forward-looking expectations of inflation. So must inflation target reflect backward-looking, forward-looking, or hybrid specifications? Assume that the delegated targets equal

(4.8a) $$\pi_t^b = \pi^a + \delta\left(\pi_{t-1} - \pi^a\right) \text{ and}$$

(4.8b) $$x_t^b = x^a + \delta\left(x_{t-1} - x^a\right),$$

The consolidated first-order condition of discretionary policy under the loss function (2.7) with targets in equations (4.8a) and (4.8b) subject to the Phillips curve (4.1) equals[30]

(4.9) $$\begin{aligned}&\kappa\left\{\left[\left(\pi_t - \pi^a\right) - \delta\left(\pi_{t-1} - \pi^a\right)\right] - \beta\delta\left[\left(E_t\pi_{t+1} - \pi^a\right) - \delta\left(\pi_t - \pi^a\right)\right]\right\} \\ &+ \lambda^b\left\{\left[\left(x_t - x^a\right) - \delta\left(x_{t-1} - x^a\right)\right] - \beta\left(\phi + \delta\right)\left[\left(E_t x_{t+1} - x^a\right) - \delta\left(x_t - x^a\right)\right]\right\} \\ &+ \lambda^b\left\{\phi\beta^2\delta\left[\left(E_t x_{t+2} - x^a\right) - \delta\left(E_t x_{t+1} - x^a\right)\right]\right\} = 0.\end{aligned}$$

Combining the first-order condition (4.9) and the Phillips curve (4.1) generates discretionary policy. A necessary condition that the discretionary policy replicates the optimal policy is that the optimal solutions (4.2a) and (4.2b) satisfy the first-order condition (4.9). Using equations (4.2a), (4.2b) and $E_t u_{t+1} = \rho u_t$ in (4.9) produces[31]

(4.10) $$\lambda^b = \frac{\left(\phi - \phi_c\right)\left(1 - \beta\rho^2\right)}{\rho\left(1 - \phi\beta\rho\right)}\lambda$$

where

[30] See equation (H.3) in Appendix H. We change the subscripts, which equal period 0 in Appendix H, to period *t* here.

[31] See equations (H.5) and (H.6) and their derivations in Appendix H.

(4.11) $$\phi_c = \frac{1-\rho}{1-\beta\rho^2} \text{ and } 0 < \phi_c < 1 \text{ for } 0 < \rho < 1.$$

To ensure $\lambda^b > 0$ requires that $\phi > \phi_c$. That is, when the Phillips curve is sufficiently backward-looking (i.e., $\phi > \phi_c$), the inflation target is backward-looking.

Now, assume that the delegated targets take

(4.12a) $$\pi_t^b = \pi^a + \left(E_t \pi_{t+1} - \pi^a\right)/\delta \text{ and}$$

(4.12b) $$x_t^b = x^a + \delta\left(x_{t-1} - x^a\right).$$

The consolidated first-order condition of discretionary policy under the loss function (2.7) with targets in equations (4.12a) and (4.12b) subject to the Phillips curve (4.1) equals[32]

(4.13) $$\begin{aligned}&\kappa\left[\left(\pi_t - \pi^a\right) - \left(E_t\pi_{t+1} - \pi^a\right)/\delta\right] \\ &\quad + \lambda^b\left\{\left[\left(x_t - x^a\right) - \delta\left(x_{t-1} - x^a\right)\right] - \beta(\delta+\phi)\left[\left(E_t x_{t+1} - x^a\right) - \delta\left(x_t - x^a\right)\right]\right\} \\ &\quad + \lambda^b\left\{\phi\beta^2\delta\left[\left(E_t x_{t+2} - x^a\right) - \delta\left(x_{t+1} - x^a\right)\right]\right\} = 0.\end{aligned}$$

Combining the first-order condition (4.13) and the Phillips curve (4.1) will generate discretionary policy. A necessary condition that the discretionary policy replicates the optimal policy is that the optimal solutions in equations (4.2a) and (4.2b) satisfy the first-order condition (4.13). Using equations (4.2a), (4.2b) and $E_t u_{t+1} = \rho u_t$ in (4.13) produces[33]

(4.14) $$\lambda^b = \frac{(\phi_c - \phi)\left(1-\beta\rho^2\right)}{\delta(1-\delta\beta\rho)(1-\phi\beta\rho)}\lambda.$$

To ensure $\lambda^b > 0$ requires that $\phi < \phi_c$. That is, when the Phillips curve is sufficiently forward-looking (i.e., $\phi < \phi_c$), the inflation target is forward-looking. Moreover, in a purely

[32] See equation (I.3) in Appendix I. We change the subscripts, which equal period 0 in Appendix I, to period *t* here.

[33] See equation (I.5) and its derivation in Appendix I.

forward-looking model (i.e., $\phi = 0$), equation (4.14) reduces to equation (3.17).

Equations (4.10) and (4.14) give the central bank weights on output stabilization under different cases of $\phi > \phi_c$ and $\phi < \phi_c$. For convenience, we define expectations as principally backward-looking if $\phi > \phi_c$, and vice versa. Then, the inflation target is backward-looking, if the Phillips curve is principally backward-looking,[34] and vice versa. That is, the inflation target reflects the principal nature of expectations in the Phillips curve. Furthermore, the critical value that divides principally backward- and forward-looking models responses to β and ρ as follows:

(4.15a) $\partial\phi_c/\partial\beta > 0$ and

(4.15b) $\partial\phi_c/\partial\rho < 0$.[35]

That is, the inflation target becomes more forward-looking, when the future is more important or the cost-push shocks are less persistent.

Now consider the central bank weight. In addition to the structural parameters (i.e., the discount factor β, social weight λ, sensitivity κ, and degree ϕ of lagged inflation), the central bank weight λ^b still depends on the cost-push shock persistence parameter ρ. We discuss the properties of λ^b for different values of ϕ. Using equation (4.10) when $\phi=1$ generates

(4.16) $$\lambda^b = \lambda .$$

In a purely backward-looking model ($\phi=1$), the central bank takes the same weight on output stabilization as society. That is, the central bank is neither weight conservative nor weight liberal.

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[34] Reversing the nature of the inflation target, optimal policy now imparts a future forecast in a backward-looking model.

[35] $\frac{\partial\phi_c}{\partial\beta} = \frac{\rho^2(1-\rho)}{\left(1-\beta\rho^2\right)^2} > 0$ and $\frac{\partial\phi_c}{\partial\rho} = \frac{\beta\left(2\rho-\rho^2\right)-1}{\left(1-\beta\rho^2\right)^2} < 0$ for $0 < \rho < 1$.

Using equation (4.10) when $\phi_c < \phi < 1$ produces

(4.17a) $\lambda^b < \lambda$ and $\partial\lambda^b / \partial\phi > 0$,

(4.17b) $\partial\lambda^b / \partial\beta < 0$,

(4.17c) $\partial\lambda^b / \partial\lambda > 0$,

(4.17d) $\partial\lambda^b / \partial\kappa = 0$, and

(4.17e) $\partial\lambda^b / \partial\rho \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ as $(1 - 2\phi\beta\rho) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$.[36]

In a principally backward-looking model ($\phi_c < \phi < 1$), the central bank must exhibit conservatism ($\lambda^b < \lambda$). Moreover, the central bank becomes less conservative, if the Phillips curve moves toward a purely backward-looking model (i.e., ϕ increasing, $\partial\lambda^b / \partial\phi > 0$), society places less importance on the future ($\partial\lambda^b / \partial\beta < 0$), and/or society places more weight on output stabilization ($\partial\lambda^b / \partial\lambda > 0$). The central bank weight, however, does not respond to the sensitivity of inflation to the output gap ($\partial\lambda^b / \partial\kappa = 0$). For the cost-push shock (4.17e), a more-persistent cost-push shock implies that the policy maker must place more or less weight on output stabilization, depending on the sign of $(1 - 2\phi\beta\rho)$. For a model with a heavily discounted future (i.e., β small) and/or little persistence (i.e., ρ small), increasing persistence leads to more weight on output stabilization.

In a principally forward-looking model ($0 < \phi < \phi_c$), λ^b equals equation (4.14) and now depends on δ, which depends on the parameters of the model. No analytical way exists to discuss

[36] See equations (J.1), (J.2), (J.3), (J.4), (J.5), and (J.6) and their derivations in Appendix J.

the relationship of the central bank weight with model parameters.[37]

In a purely forward-looking model ($\phi = 0$), however, we discuss λ^b in detail in Section 3.

5. Summary

We design central bank loss functions using the optimal solutions under commitment. The optimal solutions suggest two interesting parameters — the implied targets and the persistence parameter that governs the adjustment toward the implied targets (See Table 1). Based on implied targets and the persistence parameter as well as the nature of the expectations in the Phillips curve, we design the central bank's targets. With the designed targets, the relative weight between the targets emerges as well. Table 2 summarizes the central bank targets and weight.

Several results emerge. First, the implied targets conform to a trade-off between targets imposed by the structure of the macroeconomy (i.e., the Philips curve), and equal the long-run equilibrium values of target variables. In this sense, the implied targets are consistent with the models. Moreover, the implied targets may differ from the social targets, but they coincide when the social targets also satisfy that trade-off. The social targets are generally inconsistent with the models. Second, the government delegates to the central bank short-term, state-contingent targets, which converge to the long-run implied targets at a rate controlled by the persistence parameter. Though the delegated targets are state-contingent, they only depend on the model's structural parameters and, thus, are feasible. Third, for the Barro-Gordon model with output persistence, the correct delegated targets eliminate the constant average and state-contingent inflation biases, and a weight-liberal central bank removes the stabilization bias. Fourth, for the new-Keynesian

[37] See Appendix K.

models, delegated targets cannot eliminate the constant average and state-contingent inflation biases, until combined with the appropriate weight-liberal or –conservative central bank. Then all three biases disappear. The inflation target is forward- (backward-) looking, if the Phillips curve is principally forward- (backward-) looking. The weight may also be conservative or a liberal, depending on the model's parameters as well as the persistence of the cost-push shock. Table 3 shows how the weight specifically responds to parameters.[38] Fifth, reversing the nature of the inflation target, the optimal policy is forward-looking in a backward-looking model. This finding provides the complement to Woodford's (1999b) demonstration that optimal policy imparts inertia in a forward-looking model. We also support Woodford's finding as well, since we also find that the optimal policy is backward-looking in a forward-looking model. Sixth, the central bank can attain the delegated targets jointly on average. The attainability of the delegated targets provides the necessary ingredients to establish monetary policy credibility and accountability. Finally, the central bank, though operating with discretion, can replicate optimal policy.

[38] Table 3 only summarizes the weight response to parameters in the New Keynesian model. For the Barro-Gordon model with output persistence, see equations (2.11a, b, c, d). In addition, we do not generate analytical results as to how the weight responds to model parameters in a principally forward-looking New Keynesian model ($0<\phi<\phi_c$).

Table 1: Implied Targets and Persistence Parameters of Optimal Solutions

Model	Implied Inflation-Rate Target	Implied Output-Gap Target	Inflation-Rate Persistence	Output-Gap Persistence
Barro-Gordon Model with Output Persistence	π^*	0	0	η
New-Keynesian Model	π^a	x^a	δ	δ

Table 2: The Three Parameters of Central Bank Loss Function

Model		Inflation-Rate Target π_t^b	Output-Gap Target x_t^b	Central-Bank Weight λ^b
Barro-Gordon Model with Output Persistence		π^*	ηx_{t-1}	$\frac{\lambda}{1-\beta\eta^2}$
New-Keynesian Model	$0 \le \phi < \phi_c$ (Forward)	$\pi^a + \frac{\left(E_t\pi_{t+1} - \pi^a\right)}{\delta}$	$x^a + \delta\left(x_{t-1} - x^a\right)$	$\frac{\left(\phi_c - \phi\right)\left(1-\beta\rho^2\right)}{\delta\left(1-\delta\beta\rho\right)\left(1-\phi\beta\rho\right)}\lambda$
	$\phi_c < \phi \le 1$ (Backward)	$\pi^a + \delta\left(\pi_{t-1} - \pi^a\right)$	$x^a + \delta\left(x_{t-1} - x^a\right)$	$\frac{\left(\phi - \phi_c\right)\left(1-\beta\rho^2\right)}{\rho\left(1-\phi\beta\rho\right)}\lambda$

Table 3 Weight Response to Model Parameters in the New-Keynesian Models

ϕ=0	ϕ_c<ϕ<1	ϕ=1
$\lambda^b = \frac{1-\rho}{\delta(1-\delta\beta\rho)}\lambda$	$\lambda^b = \frac{(\phi-\phi_c)(1-\beta\rho^2)}{\rho(1-\phi\beta\rho)}\lambda$	$\lambda^b = \lambda$
$\left\{\begin{matrix} \lambda^b < \lambda, \text{if } \rho > \rho_{\lambda c} \\ \lambda^b = \lambda, \text{if } \rho = \rho_{\lambda c} \\ \lambda^b > \lambda, \text{if } \rho < \rho_{\lambda c} \end{matrix}\right\}$, where $\rho_{\lambda c} \equiv \frac{1-\delta}{1-\beta\delta^2}$	$\lambda^b < \lambda$ $\partial\lambda^b/\partial\lambda > 0$ and $\partial\lambda^b/\partial\phi > 0$	$\lambda^b = \lambda$ $\partial\lambda^b/\partial\lambda = 1$
$\partial\lambda^b/\partial\rho < 0$	$\partial\lambda^b/\partial\rho \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0$ as $(1-2\phi\beta\rho)\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0$	$\partial\lambda^b/\partial\rho = 0$
$\partial\lambda^b/\partial\beta > 0$	$\partial\lambda^b/\partial\beta < 0$	$\partial\lambda^b/\partial\beta = 0$
$\frac{\partial\lambda^b}{\partial\kappa}\begin{cases} \geq 0, \begin{cases} \text{if } \beta \leq \frac{1}{2}+\frac{\kappa^2}{\lambda} \\ \text{if } \beta > \frac{1}{2}+\frac{\kappa^2}{\lambda} \text{ and } \rho \leq \rho_{\kappa c} \end{cases} \\ < 0, \text{ if } \beta > \frac{1}{2}+\frac{\kappa^2}{\lambda} \text{ and } \rho > \rho_{\kappa c} \end{cases}$	$\partial\lambda^b/\partial\kappa = 0$	$\partial\lambda^b/\partial\kappa = 0$

Appendix A: Discretionary Policy with Persistent Output

The following specifies the problem of the central bank with discretion

(A.1)
$$\min_{\{\pi_t\}_{t=1}^{\infty}} E_0 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \frac{1}{2} \left[\left(\pi_t - \pi^* \right)^2 + \lambda^b \left(x_t - \eta x_{t-1} \right)^2 \right] \right\}$$
$$s.t. \begin{cases} x_t = \eta x_{t-1} + \alpha \left(\pi_t - \pi_t^e \right) + u_t \\ \pi_t^e = E_{t-1} \pi_t \end{cases}.$$

The Bellman equation for determining the discretionary policy equals

(A.2)
$$V(x_{t-1}) = E_{t-1} \min_{\pi_t} \left\{ \frac{1}{2} \left[\left(\pi_t - \pi^* \right)^2 + \lambda^b \left(x_t - \eta x_{t-1} \right)^2 \right] + \beta V(x_t) \right\}$$ with

(A.3)
$$V\left(x_t \right) = \gamma_0 + \gamma_1 x_t + \frac{1}{2} \gamma_2 x_t^2 .$$

Its first order condition becomes

(A.4)
$$\pi_t - \pi^* + \lambda^b \alpha \left(x_t - \eta x_{t-1} \right) + \beta \alpha \left(\gamma_1 + \gamma_2 x_t \right) = 0 .$$

Substituting equation (2.3) into equation (A.4) produces

(A.5)
$$\pi_t - \pi^* + \alpha \beta \gamma_1 + \beta \gamma_2 \alpha \eta x_{t-1} + \alpha^2 \left(\lambda^b + \beta \gamma_2 \right) \left(\pi_t - \pi_t^e \right) + \alpha \left(\lambda^b + \beta \gamma_2 \right) u_t = 0.$$

Taking expectations E_{t-1} of equation (A.5) gives

(A.6)
$$\pi_t^e = \pi^* - \alpha \beta \gamma_1 - \beta \gamma_2 \alpha \eta x_{t-1} .$$

Substituting equation (A.6) into equation (A.5) leaves

(A.7)
$$\pi_t = \pi^* - \alpha \beta \gamma_1 - \beta \gamma_2 \alpha \eta x_{t-1} - \frac{\alpha \left(\lambda^b + \beta \gamma_2 \right)}{1 + \alpha^2 \left(\lambda^b + \beta \gamma_2 \right)} u_t .$$

Substituting equations (A.6) and (A.7) into equation (2.3) results in

(A.8)
$$x_t = \eta x_{t-1} + \frac{1}{1 + \alpha^2 \left(\lambda^b + \beta \gamma_2 \right)} u_t .$$

Now, computing $E_{t-1} \left[L_t^b + \beta V\left(x_t \right) \right]$ using the solutions in equations (A.7) and (A.8) and comparing the coefficients of the result with $V\left(x_{t-1}\right)$ produces

(A.9a) $\left(1+\alpha^2\beta\gamma_2\right)\gamma_1\beta\eta=\gamma_1$ and

(A.9b) $\left(1+\alpha^2\beta\gamma_2\right)\gamma_2\beta\eta=\gamma_2$.

Therefore,

(A.10) $\gamma_1=\gamma_2=0$.

The discretion solutions in equations (A.7) and (A.8) become

(A.11a) $\pi_t=\pi^*-\frac{\alpha\lambda^b}{1+\alpha^2\lambda^b}u_t$, and

(A.11b) $x_t=\eta x_{t-1}+\frac{1}{1+\alpha^2\lambda^b}u_t$

Appendix B: Socially Optimal Solution, Purely Forward-Looking, New-Keynesian Model

The optimization problem minimizes the social intertemporal loss function (3.1) with period loss function (3.2) subject to the Phillips curve (3.3). Its Lagrangian expression equals

(B.1) $$\mathcal{L}=E_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\left[\left(\pi_t-\pi^*\right)^2+\lambda\left(x_t-x^*\right)^2\right]+\psi_t\left(\kappa x_t+\beta\pi_{t+1}+u_t-\pi_t\right)\right\}.$$

The first-order conditions equal

(B.2)
$$\frac{\partial\mathcal{L}}{\partial x_t}=E_0\left\{\beta^t\left[\lambda\left(x_t-x^*\right)+\kappa\psi_t\right]\right\}=0 \qquad \text{for } t\geq 0$$
$$\frac{\partial\mathcal{L}}{\partial\pi_0}=\left[\left(\pi_0-\pi^*\right)-\psi_0\right]=0 \qquad \text{for } t=0 .$$
$$\frac{\partial\mathcal{L}}{\partial\pi_t}=E_0\left\{\beta^t\left[\left(\pi_t-\pi^*\right)-\psi_t\right]+\beta^{t-1}\beta\psi_{t-1}\right\}=0 \qquad \text{for } t\geq 1$$

Eliminating the multipliers from the first-order conditions leads to the consolidated first-order conditions

(B.3a) $$x_0 - x^* = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right) \qquad \text{for } t = 0 \text{, and}$$

(B.3b) $$E_0 x_t - E_0 x_{t-1} = -\frac{\kappa}{\lambda}\left(E_0 \pi_t - \pi^*\right) \qquad \text{for } t \geq 1 .$$

Combining equations (B.3a) and (B.3b) with the Phillips curve (3.3) generates

(B.4a) $$\beta E_0 x_1 - b x_0 + (1-\beta)\frac{\kappa}{\lambda}\pi^* + x^* - \frac{\kappa}{\lambda}u_0 = 0 \qquad \text{for } t = 0 \text{, and}$$

(B.4b) $$\beta E_0 x_{t+1} - b E_0 x_t + E_0 x_{t-1} + (1-\beta)\frac{\kappa}{\lambda}\pi^* - \frac{\kappa}{\lambda}E_0 u_t = 0 \qquad \text{for } t \geq 1 ,$$

where

(B.5) $$b \equiv 1 + \beta + \frac{\kappa^2}{\lambda} .$$

The smaller root of the characteristic equation equals δ. That is,

(B.6) $$\beta\delta^2 - b\delta + 1 = 0 .$$

We can easily show that $0 < \delta < 1$.

Assume that the solution of equation (B.4b) takes the following form for $t \geq 1$:

(B.7) $$x_t = \delta x_{t-1} + e + f u_t .$$

Applying expectations to equation (B.7) and using $E_0 u_t = \rho E_0 u_{t-1}$ for $t \geq 1$ generates

(B.8) $$E_0 x_{t+1} = \delta E_0 x_t + e + f \rho E_0 u_t .$$

Substituting equation (B.8) into equation (B.4b) for $t \geq 1$ leads to

(B.9) $$\left(\beta\delta - b\right) E_0 x_t + E_0 x_{t-1} + (1-\beta)\frac{\kappa}{\lambda}\pi^* + \beta e + \left(\beta\rho f - \frac{\kappa}{\lambda}\right) E_0 u_t = 0 .$$

Transforming equation (B.9) and using $-1/\left(\beta\delta - b\right) = \delta$ from equation (B.6) results in

(B.10) $$E_0 x_t = \delta E_0 x_{t-1} + \delta\left[(1-\beta)\frac{\kappa}{\lambda}\pi^* + \beta e\right] + \left(\delta\beta\rho f - \frac{\kappa\delta}{\lambda}\right) E_0 u_t .$$

Comparing equation (B.10) with equation (B.7) produces

(B.11a) $$e=\frac{\kappa\delta\left(1-\beta\right)}{\lambda\left(1-\beta\delta\right)}\pi^{*}$$ for $t\geq 1$ and

(B.11b) $$f=-\frac{\kappa\delta}{\lambda\left(1-\delta\beta\rho\right)}$$ for $t\geq 1$.

Applying expectations E_0 to equation (B.7) for t=1 generates

(B.12) $$\begin{aligned} E_0x_1 &= \delta x_0+e+fE_0u_1=\delta x_0+e+f\rho u_0 \\ &= \delta x_0+\frac{\kappa\delta\left(1-\beta\right)}{\lambda\left(1-\beta\delta\right)}\pi^{*}-\frac{\kappa\delta\rho}{\lambda\left(1-\delta\beta\rho\right)}u_0 \end{aligned}.$$

Substituting equation (B.12) into equation (B.4a) for t=0 leads to

(B.13) $$x_0=\delta\left(\frac{\kappa}{\lambda}\frac{1-\beta}{1-\beta\delta}\pi^{*}+x^{*}\right)-\frac{\kappa\delta}{\lambda\left(1-\delta\beta\rho\right)}u_0=\delta x^{*}+e+fu_0.$$

Summarily, the socially optimal solution of the output gap equals

(B.14a) $$x_0=\delta x^{*}+e+fu_0$$ for $t=0$, and

(B.14b) $$x_t=\delta x_{t-1}+e+fu_t$$ for $t\geq 1$,

where e and f satisfy equations (B.11a) and (B.11b).

To interpret the constant e in equations (B.14a) and (B.14b), we write $e=[e/\left(1-\delta\right)-\delta e/(1-\delta)]$, where $x^{a}\equiv e/\left(1-\delta\right)$, which equals the implied output gap target. Thus, equations (B.14a) and (B.14b) become

(B.15a) $$(x_0-x^{a})=\delta(x^{*}-x^{a})+fu_0$$ for $t=0,$ and

(B.15b) $$(x_t-x^{a})=\delta(x_{t-1}-x^{a})+fu_0$$ for $t\geq 1$.

Using equations (B.5) and (B.6), we can show that

(B.16) $$\left(1-\beta\delta\right)\left(1-\delta\right)=\beta\delta^{2}+1-\delta-\beta\delta=b\delta-\delta-\beta\delta=\frac{\kappa^{2}}{\lambda}\delta.$$

Thus, $x^a \equiv \frac{e}{1-\delta} = \frac{\kappa\delta(1-\beta)}{\lambda(1-\beta\delta)(1-\delta)}\pi^* = \frac{1-\beta}{\kappa}\pi^*$. That is,

(B.17) $$x^a \equiv \frac{1-\beta}{\kappa}\pi^*.$$

Now, we can solve for π_t as follows. Using equation (B.7) for $t\geq 2$ generates

(B.18a) $$E_0 x_t = \delta E_0 x_{t-1} + e + fE_0 u_t \text{, and}$$

(B.18b) $$E_0 x_{t-1} = \delta E_0 x_{t-2} + e + \frac{f}{\rho} E_0 u_t .$$

Using equations (B.18a), (B.18b), and (B.11b) leads to

(B.19) $$E_0 x_t - E_0 x_{t-1} = \delta\left(E_0 x_{t-1} - E_0 x_{t-2}\right) + \frac{\kappa\delta(1-\rho)}{\lambda\rho(1-\delta\beta\rho)} E_0 u_t \qquad \text{for } t \geq 2 .$$

Substituting equation (B.3b) for $t\geq 2$ into equation (B.19) gives

(B.20) $$\left(E_0 \pi_t - \pi^*\right) = \delta\left(E_0 \pi_{t-1} - \pi^*\right) - \frac{\delta(1-\rho)}{\rho(1-\delta\beta\rho)} E_0 u_t \qquad \text{for } t \geq 2 .$$

Using equations (B.15a) and (B.15b) for t=1 and $u_0 = \frac{1}{\rho} E_0 u_1$ gives

(B.21) $$E_0 x_1 - x_0 = \delta\left(x_0 - x^*\right) + \frac{\kappa\delta(1-\rho)}{\lambda\rho(1-\delta\beta\rho)} E_0 u_1 .$$

According consolidated first-order conditions in equations (B.3a) and (B.3b), we observe that $x_0 - x^* = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right)$ and $E_0 x_1 - x_0 = -\frac{\kappa}{\lambda}\left(E_0 \pi_1 - \pi^*\right)$. Substituting these results into (B.21) leads to

(B.22) $$\left(E_0 \pi_1 - \pi^*\right) = \delta\left(\pi_0 - \pi^*\right) - \frac{\delta(1-\rho)}{\rho(1-\delta\beta\rho)} E_0 u_1 .$$

Using equations (B.3a) and (B.15a) for t=0 produces

(B.23) $$\left(\pi_0 - \pi^*\right) = \frac{\lambda}{\kappa}(1-\delta)\left(x^* - x^a\right) + \frac{\delta}{(1-\delta\beta\rho)} u_0 .$$

Combining equations (B.20), (B.22), and (B.23) gives

(B.24a) $$\left(\pi_0 - \pi^*\right) = \frac{\lambda}{\kappa}(1-\delta)\left(x^* - x^a\right) + \frac{\delta}{\left(1-\delta\beta\rho\right)}u_0$$ for $t = 0$, and

(B.24b) $$\left(\pi_t - \pi^*\right) = \delta\left(\pi_{t-1} - \pi^*\right) - \frac{\delta(1-\rho)}{\rho\left(1-\delta\beta\rho\right)}u_t$$ for $t \geq 1$.

Appendix C: Persistence Response to Structural Parameters in a Purely Forward-Looking, New-Keynesian Model

The characteristic equation equals the following:

(C.1) $$\beta\delta^2 - b\delta + 1 = 0.$$

The smaller root of the characteristic equation equals δ. That is,

(C.2) $$\delta = \frac{b - \sqrt{b^2 - 4\beta}}{2\beta}.$$

We can easy verify that

(C.3) $$b^2 - 4\beta > 0 \text{ and } 0 < \delta < 1.$$

Transforming equation (C.2) results in

(C.4) $$2\beta\delta - b = -\sqrt{b^2 - 4\beta}.$$

Differentiating δ with respect to β in the characteristic equation (C.1) leads to

(C.5) $$\delta^2 + \left(2\beta\delta - b\right)\frac{\partial\delta}{\partial\beta} - \delta = 0.$$

Transforming equation (C.5) gives

(C.6) $$\frac{\partial\delta}{\partial\beta} = \frac{\delta(1-\delta)}{2\beta\delta - b}.$$

Substituting equation (C.4) into equation (C.6) produces

(C.7) $$\frac{\partial \delta}{\partial \beta} = -\frac{\delta(1-\delta)}{\sqrt{b^2 - 4\beta}}.$$

Noting equation (C.3), we can see that

(C.8) $$\frac{\partial \delta}{\partial \beta} < 0.$$

Differentiating δ with respect to λ in the characteristic equation (C.1) gets

(C.9) $$2\beta\delta \frac{\partial \delta}{\partial \lambda} + \frac{\kappa^2}{\lambda^2}\delta - b\frac{\partial \delta}{\partial \lambda} = 0.$$

Transforming equation (C.9) generates

(C.10) $$\frac{\partial \delta}{\partial \lambda} = -\frac{\kappa^2 \delta}{\lambda^2 (2\beta\delta - b)}.$$

Substituting equation (C.4) into equation (C.10) results in

(C.11) $$\frac{\partial \delta}{\partial \lambda} = \frac{\kappa^2 \delta}{\lambda^2 \sqrt{b^2 - 4\beta}} > 0.$$

Differentiating δ with respect to κ in the characteristic equation (C.1) leads to

(C.12) $$2\beta\delta \frac{\partial \delta}{\partial \kappa} - \frac{2\kappa}{\lambda}\delta - b\frac{\partial \delta}{\partial \kappa} = 0.$$

Transforming equation (C.12) produces

(C.13) $$\frac{\partial \delta}{\partial \kappa} = \frac{2\kappa\delta}{\lambda(2\beta\delta - b)} < 0.$$

Appendix D: Determining λ^b with $\pi_t^b = \pi^* + \delta(\pi_{t-1} - \pi^*)$ in a Purely Forward-Looking, New-Keynesian Model

The central bank operates with discretion. That is, the central bank always re-minimizes each period, subject to the Phillips curve (3.3), the expectation of the intertemporal loss function

$$\text{(D.1)} \qquad \mathbb{L}^b = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t^b\right),$$

with the period loss function (2.7) and targets defined in equations (3.11a) and (3.11b). The Lagrangian expression of the problem equals

$$\text{(D.2)} \qquad \mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\left(\pi_t - \pi^* - \delta\left(\pi_{t-1} - \pi^*\right)\right)^2 + \lambda^b\left(x_t - x^a - \delta\left(x_{t-1} - x^a\right)\right)^2\right] \\ +\psi_t\left(\kappa x_t + \beta\pi_{t+1} + u_t - \pi_t\right) \end{array} \right\}.$$

The first-order conditions with respect to π and x equal the following:

$$\frac{\partial\mathcal{L}}{\partial x_t} = E_0\left\{ \begin{array}{l} \beta^t\left[\lambda^b\left(x_t - x^a - \delta\left(x_{t-1} - x^a\right)\right) + \kappa\psi_t\right] \\ \qquad -\beta^{t+1}\delta\lambda^b\left(x_{t+1} - x^a - \delta\left(x_t - x^a\right)\right) \end{array} \right\} = 0 \qquad \text{for } t = 0,1,\ldots$$

$$\text{(D.3)} \qquad \frac{\partial\mathcal{L}}{\partial\pi_0} = E_0\left\{ \left[\begin{array}{l} \left(\pi_0 - \pi^* - \delta\left(\pi_{-1} - \pi^*\right)\right) - \psi_0 \\ \qquad -\beta\delta\left(\pi_1 - \pi^* - \delta\left(\pi_0 - \pi^*\right)\right) \end{array} \right] \right\} = 0 \qquad \text{for } t = 0$$

$$\frac{\partial\mathcal{L}}{\partial\pi_t} = E_0\left\{ \begin{array}{l} \beta^t\left[\left(\pi_t - \pi^* - \delta\left(\pi_{t-1} - \pi^*\right)\right) - \psi_t\right] \\ \qquad -\beta^{t+1}\delta\left(\pi_{t+1} - \pi^* - \delta\left(\pi_t - \pi^*\right)\right) + \beta^{t-1}\beta\psi_{t-1} \end{array} \right\} = 0 \quad \text{for } t = 1,2,\ldots$$

As the central bank re-makes policy each period, eliminating the multiplier ψ_0 leads to the consolidated first-order condition for t=0

$$\text{(D.4)} \qquad \begin{array}{l} \kappa\left\{\left[\left(\pi_0 - \pi^*\right) - \delta\left(\pi_{-1} - \pi^*\right)\right] - \beta\delta\left[\left(E_0\pi_1 - \pi^*\right) - \delta\left(\pi_0 - \pi^*\right)\right]\right\} \\ \quad + \lambda^b\left\{\left[\left(x_0 - x^a\right) - \delta\left(x_{-1} - x^a\right)\right] - \beta\delta\left[\left(E_0 x_1 - x^a\right) - \delta\left(x_0 - x^a\right)\right]\right\} = 0 \end{array}.$$

As we require that discretionary policy replicates the optimal policy, the optimal solutions in equations (3.7a) and (3.7b) from timeless perspective must satisfy equation (D.4). Using equations (3.7a) and (3.7b) for t=0,1 and $E_0 u_1 = \rho u_0$ results in

$$\text{(D.5a)} \qquad \pi_0 - \pi^* - \delta\left(\pi_{-1} - \pi^*\right) = -\frac{\delta(1-\rho)}{\rho(1-\delta\beta\rho)}u_0,$$

(D.5b) $$E_0\pi_1 - \pi^* - \delta\left(\pi_0 - \pi^*\right) = -\frac{\delta\left(1-\rho\right)}{\left(1-\delta\beta\rho\right)}u_0,$$

(D.5c) $$x_0 - x^a - \delta\left(x_{-1} - x^a\right) = -\frac{\kappa\delta}{\lambda\left(1-\delta\beta\rho\right)}u_0, \text{ and}$$

(D.5d) $$E_0 x_1 - x^a - \delta\left(x_0 - x^a\right) = -\frac{\kappa\delta\rho}{\lambda\left(1-\delta\beta\rho\right)}u_0.$$

Substituting equations (D.5a), (D.5b), (D.5c), and (D.5d) into equation (D.4) implies that λ^b equals the following:

(D.6) $$\lambda^b = -\frac{\left(1-\rho\right)}{\rho}\lambda.$$

Appendix E: Determining $\boldsymbol{\lambda^b}$ with $\pi_t^b = \pi^* + \left(E_t\pi_{t+1} - \pi^*\right)/\delta$ in a Purely Forward-Looking, New-Keynesian Model

The problem follows the same path as in Appendix D, except that $\pi_t^b = \pi^* + \left(E_t\pi_{t+1} - \pi^*\right)/\delta$ rather than $\pi_t^b = \delta\pi_{t-1}$. The Lagrangian expression of the problem equals

(E.1) $$\mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t\left\{\begin{array}{c}\frac{1}{2}\left[\left(\pi_t - \pi^* - \left(\pi_{t+1} - \pi^*\right)/\delta\right)^2 + \lambda^b\left(x_t - x^a - \delta\left(x_{t-1} - x^a\right)\right)^2\right] \\ +\psi_t\left(\kappa x_t + \beta\pi_{t+1} + u_t - \pi_t\right)\end{array}\right\}.$$

As the central bank operates with discretion, we only need to calculate the first-order conditions regarding π_0 and x_0 to determine λ^b.

(E.2) $$\frac{\partial\mathcal{L}}{\partial\pi_0} = E_0\left[\left(\pi_0 - \pi^* - \left(\pi_1 - \pi^*\right)/\delta\right) - \psi_0\right] = 0; \text{ and}$$
$$\frac{\partial\mathcal{L}}{\partial x_0} = E_0\left\{\left[\lambda^b\left(x_0 - x^a - \delta\left(x_{-1} - x^a\right)\right) + \kappa\psi_0\right] - \beta\delta\lambda^b\left(x_1 - x^a - \delta\left(x_0 - x^a\right)\right)\right\} = 0.$$

Eliminating the multiplier ψ_0 results in

(E.3)
$$\kappa\left[\left(\pi_0-\pi^*\right)-\left(E_0\pi_1-\pi^*\right)/\delta\right] + \lambda^b\left\{\left[\left(x_0-x^a\right)-\delta\left(x_{-1}-x^a\right)\right]-\beta\delta\left[\left(E_0x_1-x^a\right)-\delta\left(x_0-x^a\right)\right]\right\}=0 .$$

As we require that discretionary policy replicates the optimal policy, the optimal solutions in equations (3.7a) and (3.7b) from the timeless perspective must satisfy equation (E.3). Using equations (3.7a) and (3.7b) for t=0,1 and $E_0u_1=\rho u_0$ leads to

(E.4a)
$$\pi_0-\pi^*-\left(E_0\pi_1-\pi^*\right)/\delta=\frac{1-\rho}{1-\delta\beta\rho}u_0 ,$$

(E.4b)
$$x_0-x^a-\delta\left(x_{-1}-x^a\right)=-\frac{\kappa\delta}{\lambda\left(1-\delta\beta\rho\right)}u_0 , \text{ and}$$

(E.4c)
$$E_0x_1-x^a-\delta\left(x_0-x^a\right)=-\frac{\kappa\delta\rho}{\lambda\left(1-\delta\beta\rho\right)}u_0 .$$

Substituting equations (E.4a), (E.4b), and (E.4c) into equation (E.3) gives

(E.5)
$$\lambda^b=\frac{1-\rho}{\delta\left(1-\delta\beta\rho\right)}\lambda .$$

Appendix F: Weight Response to Structural Parameters in a Purely Forward-Looking, New-Keynesian Model

The weight in the central bank loss function equals

(F.1)
$$\lambda^b=\frac{1-\rho}{\delta\left(1-\delta\beta\rho\right)}\lambda .$$

Differentiating λ^b with respect to ρ produces

(F.2)
$$\frac{\partial\lambda^b}{\partial\rho}=-\frac{1-\beta\delta}{\delta\left(1-\delta\beta\rho\right)^2}\lambda<0 .$$

Differentiating λ^b with respect to β generates

(F.3) $$\frac{\partial \lambda^b}{\partial \beta} = -\frac{\lambda(1-\rho)}{\delta^2(1-\delta\beta\rho)^2}\left[(1-2\delta\beta\rho)\frac{\partial \delta}{\partial \beta} - \delta^2\rho\right].$$

Substituting equation (C.6) into equation (F.3) results in

(F.4) $$\frac{\partial \lambda^b}{\partial \beta} = \lambda(1-\rho)\frac{(1-\delta)(1-2\delta\beta\rho) - \delta\rho(2\beta\delta - b)}{-\delta(1-\delta\beta\rho)^2(2\beta\delta - b)}.$$

Noting equation (C.4), we determine that

(F.5) $$-\delta(1-\delta\beta\rho)^2(2\beta\delta - b) > 0.$$

Focusing on the numerator of equation (F.4), we determine that

(F.6) $$(1-\delta)(1-2\delta\beta\rho) - \delta\rho(2\beta\delta - b) = (1-\delta) + (b - 2\beta)\delta\rho > 0.$$

According to equations (F.5) and (F.6), we can show that

(F.7) $$\frac{\partial \lambda^b}{\partial \beta} > 0.$$

Differentiating λ^b with respect to λ and substituting equation (C.10) into the result leads to

(F.8) $$\frac{\partial \lambda^b}{\partial \lambda} = \frac{(1-\rho)}{\delta(1-\delta\beta\rho)^2}\left[(1-\delta\beta\rho) + \frac{\kappa^2(1-2\delta\beta\rho)}{\lambda(2\beta\delta - b)}\right].$$

We cannot determine the sign of $\partial\lambda^b/\partial\lambda$, since it depends on the values of the parameters. Differentiating λ^b with respect to κ and substituting equation (C.13) into the result produces

(F.9) $$\frac{\partial \lambda^b}{\partial \kappa} = \frac{2\kappa(1-\rho)}{-(2\beta\delta - b)\delta(1-\delta\beta\rho)^2}(1-2\delta\beta\rho).$$

Noting equation (C.4), we know that the sign of $\frac{2\kappa(1-\rho)}{-(2\beta\delta - b)\delta(1-\delta\beta\rho)^2}$ is positive. So the sign of $\frac{\partial \lambda^b}{\partial \kappa}$ depends on the sign of $(1-2\delta\beta\rho)$.

Appendix G: Socially Optimal Solution in a Hybrid New-Keynesian Model

The optimization problem minimizes the social intertemporal loss function (3.1) with period loss function (3.2) subject to the Phillips curve (4.1). Its Lagrangian expression equals the following:

(G.1) $$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\left(x_t - x^*\right)^2 \right] + \psi_t \left[\kappa x_t + \phi \pi_{t-1} + \left(1-\phi\right)\beta \pi_{t+1} + u_t - \pi_t \right] \right\}.$$

The first-order conditions equal

(G.2)
$$\frac{\partial \mathcal{L}}{\partial x_t} = E_0 \left\{ \beta^t \left[\lambda\left(x_t - x^*\right) + \kappa \psi_t \right] \right\} = 0 \qquad \text{for } t \ge 0$$
$$\frac{\partial \mathcal{L}}{\partial \pi_0} = E_0 \left[\left(\pi_0 - \pi^*\right) - \psi_0 + \beta\phi\psi_1 \right] = 0 \qquad \text{for } t = 0 .$$
$$\frac{\partial \mathcal{L}}{\partial \pi_t} = E_0 \left\{ \beta^t \left[\left(\pi_t - \pi^*\right) - \psi_t \right] + \beta^{t+1}\phi\psi_{t+1} + \beta^{t-1}\left(1-\phi\right)\beta\psi_{t-1} \right\} = 0 \qquad \text{for } t \ge 1$$

Eliminating the multipliers from equation (G.2) gives the consolidated first-order conditions as follows:

(G.3a) $$\left(x_0 - x^*\right) - \phi\beta\left(E_0 x_1 - x^*\right) = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right) \qquad \text{for } t = 0 \text{, and}$$

(G.3b) $$\left(E_0 x_t - x^*\right) - \phi\beta\left(E_0 x_{t+1} - x^*\right) - \left(1-\phi\right)\left(E_0 x_{t-1} - x^*\right) = -\frac{\kappa}{\lambda}\left(E_0 \pi_t - \pi^*\right) \qquad \text{for } t \ge 1$$

We work backward to solve equations (G.3a) and (G.3b). That is, we first solve $\{\pi_t, x_t\}$ for $t \geq 1$, then $\{\pi_0, x_0\}$. Combining equations (G.3b) with the Phillips curve (4.1) produces

(G.4) $$a\beta^2 E_0 x_{t+2} - \beta E_0 x_{t+1} + b E_0 x_t - E_0 x_{t-1} + a E_0 x_{t-2} - \left(1-\phi\right)\left(1-\beta\right)\left[\frac{\kappa}{\lambda}\pi^* + \phi\left(1-\beta\right)x^* \right] + \frac{\kappa}{\lambda} E_0 u_t = 0 \qquad \text{for } t \ge 1,$$

where

(G.5a) $$a \equiv \phi\left(1-\phi\right)$$

(G.5b) $$b \equiv 1 + \frac{\kappa^2}{\lambda} + \phi^2 \beta + (1-\phi)^2 \beta = 1 + \beta + \frac{\kappa^2}{\lambda} - 2a\beta \text{, and}$$

(G.5c) $$x_{-1} \equiv 0 .$$

Assume that δ equals a root of the characteristic equation

(G.6) $$a\beta^2\delta^4 - \beta\delta^3 + b\delta^2 - \delta + a = 0 .$$

Assume that the solution of equation (G.4) for $t \geq 1$ takes the following form:

(G.7) $$x_t = \delta x_{t-1} + e + fu_t .$$

Using equation (G.7) and $E_0 u_t = \rho E_0 u_{t-1}$ leads to

(G.8a) $$E_0 x_t = \delta E_0 x_{t-1} + e + f\rho E_0 u_{t-1} ,$$

(G.8b) $$E_0 x_{t+1} = \delta^2 E_0 x_{t-1} + (\delta + 1)e + (\delta + \rho) f \rho E_0 u_{t-1} \text{, and}$$

(G.8c) $$E_0 x_{t+2} = \delta^3 E_0 x_{t-1} + (\delta^2 + \delta + 1)e + (\delta^2 + \delta\rho + \rho^2) f \rho E_0 u_{t-1} .$$

Substituting equations (G.8a), (G.8b), (G.8c), and $E_0 u_t = \rho E_0 u_{t-1}$ into equation (G.4) results in

(G.9) $$\begin{aligned}&\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right) E_0 x_{t-1} + aE_0 x_{t-2} + \left(df + \frac{\kappa}{\lambda}\right)\rho E_0 u_{t-1} \\ &\qquad + ce - (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] = 0,\end{aligned}$$

where

(G.10a) $$c \equiv a\beta^2\left(\delta^2 + \delta + 1\right) - \beta(\delta + 1) + b \quad \text{and}$$

(G.10b) $$d \equiv a\beta^2\left(\delta^2 + \delta\rho + \rho^2\right) - \beta(\delta + \rho) + b .$$

Transforming equation (G.9) and noting that $-1/\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right) = \delta/a$ from equation (G.6) produces

(G.11)
$$E_0 x_{t-1} = \delta E_0 x_{t-2} + \frac{\delta}{a}\left\{ce - (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right]\right\} + \frac{\delta}{a}\left(df + \frac{\kappa}{\lambda}\right)\rho E_0 u_{t-1}.$$

Comparing equation (G.11) with equation (G.7) implies

(G.12a)
$$e = \frac{\delta(1-\phi)(1-\beta)}{\delta c - a}\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] \text{ and}$$

(G.12b)
$$f = -\frac{\kappa\delta\rho}{\lambda(\delta\rho d - a)}.$$

Similarly, to interpret the constant e in equation (G.7), we compute the implied output gap target

(G.13)
$$x^a \equiv \frac{e}{1-\delta} = \frac{\delta(1-\phi)(1-\beta)}{(\delta c - a)(1-\delta)}\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right].$$

Note that $x^a = 0$, when $\phi = 1$.

Compute

(G.14)
$$\begin{aligned}\delta c - a &= \delta\left[a\beta^2\left(\delta^2 + \delta + 1\right) - \beta(\delta + 1) + b\right] - a \\ &= a\beta^2\left(\delta^3 + \delta^2 + \delta\right) - \beta\left(\delta^2 + \delta\right) + b\delta - a \\ &= a\beta^2\delta^3 + a\beta^2\delta^2 + a\beta^2\delta - \beta\delta^2 - \beta\delta + b\delta - a \\ &= \left(a\beta^2\delta^3 - \beta\delta^2 + b\delta\right) + a\beta^2\delta^2 + a\beta^2\delta - \beta\delta - a\end{aligned} \quad \text{and}$$

(G.15)
$$\begin{aligned}(\delta c - a)\delta &= \left[\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta\right) + a\beta^2\delta^2 + a\beta^2\delta - \beta\delta - a\right]\delta \\ &= \left(a\beta^2\delta^4 - \beta\delta^3 + b\delta^2\right) + a\beta^2\delta^3 + a\beta^2\delta^2 - \beta\delta^2 - a\delta\end{aligned}.$$

Noting that $a\beta^2\delta^4 - \beta\delta^3 + b\delta^2 = \delta - a$ from equation (G.6), (G.15) becomes

(G.16)
$$(\delta c - a)\delta = \delta - a + a\beta^2\delta^3 + a\beta^2\delta^2 - \beta\delta^2 - a\delta.$$

Subtracting equation (G.16) from equation (G.14) gives

(G.17) $$\left(\delta c-a\right)\left(1-\delta\right)=\left(\delta c-a\right)-\left(\delta c-a\right)\delta=\left(a\beta^2+b-\beta-1+a\right)\delta.$$

Substituting equation (G.5b) into equation (G.17) produces

(G.18) $$a\beta^2+b-\beta-1+a=\frac{\kappa^2}{\lambda}+a\left(1-\beta\right)^2.$$

The implied output target equal

(G.19) $$\begin{aligned} x^a \equiv \frac{e}{1-\delta} &= \left(1-\phi\right)\left(1-\beta\right)\left[\frac{\kappa\pi^* + \lambda\phi\left(1-\beta\right)x^*}{\kappa^2+\lambda a\left(1-\beta\right)^2}\right] \\ &= \frac{\left(1-\phi\right)\left(1-\beta\right)\lambda}{\kappa^2+\lambda a\left(1-\beta\right)^2}\left[\frac{\kappa}{\lambda}\pi^*+\phi\left(1-\beta\right)x^*\right]. \end{aligned}$$

Using equation (G.7) for $t\geq 1$ leads to

(G.20a) $$E_0 x_t = \delta E_0 x_{t-1} + e + f E_0 u_t,$$

(G.20b) $$E_0 x_{t+1} = \delta E_0 x_t + e + f\rho E_0 u_t, \text{ and}$$

(G.20c) $$E_0 x_{t-1} = \delta E_0 x_{t-2} + e + \frac{f}{\rho} E_0 u_t.$$

Using equations (G.20a), (G.20b), and (G.20c) yields

(G.21) $$\begin{aligned} E_0x_t-\phi\beta E_0x_{t+1}-\left(1-\phi\right)E_0x_{t-1} &= \delta\left[E_0x_{t-1}-\phi\beta E_0x_t-\left(1-\phi\right)E_0x_{t-2}\right] \\ &\quad +\phi\left(1-\beta\right)e+\left[\phi\left(1-\beta\rho^2\right)-\left(1-\rho\right)\right]\frac{f}{\rho}E_0u_t. \end{aligned}$$

Transforming equations (G.3b) gives

(G.22) $$E_0x_t-\phi\beta E_0x_{t+1}-\left(1-\phi\right)E_0x_{t-1}=\phi\left(1-\beta\right)x^*-\frac{\kappa}{\lambda}\left(E_0\pi_t-\pi^*\right) \quad \text{for } t\geq 1.$$

Substituting equations (G.22) (for $t\geq 1$) and (G.12b) into equation (G.21) generates

(G.23) $$\begin{aligned} E_0\pi_t &= \delta E_0\pi_{t-1}+\left(1-\delta\right)\left[\pi^*+\frac{\lambda}{\kappa}\phi\left(1-\beta\right)x^*\right]-\frac{\lambda}{\kappa}\phi\left(1-\beta\right)e \\ &\quad +\left[\phi\left(1-\beta\rho^2\right)-\left(1-\rho\right)\right]\frac{\delta}{\left(\delta\rho d-a\right)}E_0u_t. \end{aligned}$$

The constant term in equation (G.23) equals

(G.24)
$$\begin{aligned}&(1-\delta)\left[\pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^*\right] - \frac{\lambda}{\kappa}\phi(1-\beta)e \\ &= (1-\delta)\left[\pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda}{\kappa}\phi(1-\beta)x^a\right].\end{aligned}$$

Denote

(G.25)
$$\pi^a \equiv \pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda}{\kappa}\phi(1-\beta)x^a = \pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)(x^* - x^a).$$

Thus, if $\phi = 0$, then $\pi^a = \pi^*$ and if $\phi = 1$, then $\pi^a = \pi^* + \frac{\lambda}{\kappa}(1-\beta)x^*$, since $x^a = 0$.

Equation (G.23) becomes

(G.26)
$$E_0(\pi_t - \pi^a) = \delta(E_0\pi_{t-1} - \pi^a) + \left[\phi(1-\beta\rho^2) - (1-\rho)\right]\frac{\delta}{(\delta\rho d - a)}E_0 u_t.$$

Now compute π^a. Substituting equation (G.19) into equation (G.25) produces

$$\begin{aligned}\pi^a &\equiv \pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda}{\kappa}\phi(1-\beta)(1-\phi)(1-\beta)\left[\frac{\kappa\pi^* + \lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2}\right] \\ &= \pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda}{\kappa}a(1-\beta)^2\left[\frac{\kappa\pi^* + \lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2}\right] \\ &= \pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda}{\kappa}a(1-\beta)^2\frac{\kappa\pi^*}{\kappa^2 + \lambda a(1-\beta)^2} - \frac{\lambda}{\kappa}a(1-\beta)^2\frac{\lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2} \\ &= \pi^* - \frac{\lambda a(1-\beta)^2}{\kappa^2 + \lambda a(1-\beta)^2}\pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^* - \frac{\lambda a(1-\beta)^2}{\kappa^2 + \lambda a(1-\beta)^2}\frac{\lambda}{\kappa}\phi(1-\beta)x^* \\ &= \left[1 - \frac{\lambda a(1-\beta)^2}{\kappa^2 + \lambda a(1-\beta)^2}\right]\pi^* + \left[1 - \frac{\lambda a(1-\beta)^2}{\kappa^2 + \lambda a(1-\beta)^2}\right]\frac{\lambda}{\kappa}\phi(1-\beta)x^* \\ &= \frac{\kappa^2}{\kappa^2 + \lambda a(1-\beta)^2}\left[\pi^* + \frac{\lambda}{\kappa}\phi(1-\beta)x^*\right]\end{aligned}.$$

That is,

(G.27) $$\pi^a = \frac{\kappa\lambda}{\kappa^2 + \lambda a(1-\beta)^2}\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right].$$

In sum, the optimal paths $\{\pi_t, x_t\}$ for $t \geq 1$ equal

(G.28a) $$\pi_t = \pi^a + \delta\left(\pi_{t-1} - \pi^a\right) + \left[\phi\left(1-\beta\rho^2\right) - (1-\rho)\right]\frac{\delta}{(\delta\rho d - a)}u_t, \text{ and}$$

(G.28b) $$x_t = x^a + \delta\left(x_{t-1} - x^a\right) - \frac{\kappa\delta\rho}{\lambda(\delta\rho d - a)}u_t,$$

where π^a and x^a are defined by equations (G.27) and (G.19). As we adopt the timeless optimal policy, we do not compute the policy for t=0.

Appendix H: Determining λ^b with $\pi_t^b = \pi^a + \delta\left(\pi_{t-1} - \pi^a\right)$ in a Hybrid New-Keynesian Model

The Lagrangian expression of the problem equals

(H.1) $$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\left(\pi_t - \pi^a - \delta\left(\pi_{t-1} - \pi^a\right)\right)^2 + \lambda^b\left(x_t - x^a - \delta\left(x_{t-1} - x^a\right)\right)^2\right] \\ + \psi_t\left[\kappa x_t + \phi\pi_{t-1} + (1-\phi)\beta\pi_{t+1} + u_t - \pi_t\right] \end{array} \right\}.$$

As the central bank operates with discretion, we only need to calculate the first-order conditions with respect to π_0, x_0, and x_1

(H.2)
$$\frac{\partial\mathcal{L}}{\partial\pi_0} = E_0 \left\{ \begin{array}{l} \left[\left(\pi_0 - \pi^a\right) - \delta\left(\pi_{-1} - \pi^a\right)\right] - \psi_0 \\ \qquad - \beta\delta\left[\left(\pi_1 - \pi^a\right) - \delta\left(\pi_0 - \pi^a\right)\right] + \beta\phi\psi_1 \end{array} \right\} = 0,$$

$$\frac{\partial\mathcal{L}}{\partial x_0} = E_0 \left\{ \begin{array}{l} \lambda^b\left[\left(x_0 - x^a\right) - \delta\left(x_{-1} - x^a\right)\right] + \kappa\psi_0 \\ \qquad - \beta\delta\lambda^b\left[\left(x_1 - x^a\right) - \delta\left(x_0 - x^a\right)\right] \end{array} \right\} = 0, \text{ and}$$

$$\frac{\partial\mathcal{L}}{\partial x_1} = E_0 \left\{ \begin{array}{l} \beta\left\{\lambda^b\left[\left(x_1 - x^a\right) - \delta\left(x_0 - x^a\right)\right] + \kappa\psi_1\right\} \\ \qquad - \beta^2\delta\lambda^b\left[\left(x_2 - x^a\right) - \delta\left(x_1 - x^a\right)\right] \end{array} \right\} = 0.$$

Eliminating multipliers ψ_0 and ψ_1 yields

$$\begin{aligned}\text{(H.3)}\quad & \kappa\left\{\left[\left(\pi_0-\pi^a\right)-\delta\left(\pi_{-1}-\pi^a\right)\right]-\beta\delta\left[\left(E_0\pi_1-\pi^a\right)-\delta\left(\pi_0-\pi^a\right)\right]\right\} \\ & +\lambda^b\left\{\left[\left(x_0-x^a\right)-\delta\left(x_{-1}-x^a\right)\right]-\beta\left(\phi+\delta\right)\left[\left(E_0x_1-x^a\right)-\delta\left(x_0-x^a\right)\right]\right\} \\ & +\lambda^b\left\{\phi\beta^2\delta\left[\left(E_0x_2-x^a\right)-\delta\left(E_0x_1-x^a\right)\right]\right\}=0.\end{aligned}$$

Using equations (4.2a) and (4.2b) for t=0,1,2, $E_0u_1=\rho u_0$ and $E_0u_2=\rho^2u_0$ results in

(H.4a) $$\left(\pi_0-\pi^a\right)-\delta\left(\pi_{-1}-\pi^a\right)=\left[\phi\left(1-\beta\rho^2\right)-\left(1-\rho\right)\right]\frac{\delta}{\left(\delta\rho d-a\right)}u_0,$$

(H.4b) $$\left(E_0\pi_1-\pi^a\right)-\delta\left(\pi_0-\pi^a\right)=\left[\phi\left(1-\beta\rho^2\right)-\left(1-\rho\right)\right]\frac{\delta}{\left(\delta\rho d-a\right)}\rho u_0,$$

(H.4c) $$\left(x_0-x^a\right)-\delta\left(x_{-1}-x^a\right)=-\frac{\kappa\delta\rho}{\lambda\left(\delta\rho d-a\right)}u_0,$$

(H.4d) $$\left(E_0x_1-x^a\right)-\delta\left(x_0-x^a\right)=-\frac{\kappa\delta\rho}{\lambda\left(\delta\rho d-a\right)}\rho u_0, \text{ and}$$

(H.4e) $$\left(E_0x_2-x^a\right)-\delta\left(E_0x_1-x^a\right)=-\frac{\kappa\delta\rho}{\lambda\left(\delta\rho d-a\right)}\rho^2u_0.$$

Substituting equations (H.4a), (H.4b), (H.4c), (H.4d), and (H.4e) into equation (H.3) leads to λ^b as follows:

(H.5) $$\lambda^b=\frac{\left(\phi-\phi_c\right)\left(1-\beta\rho^2\right)}{\rho\left(1-\phi\beta\rho\right)}\lambda$$

where

(H.6) $$\phi_c=\frac{1-\rho}{1-\beta\rho^2}.$$

Appendix I: Determining λ^b with $\pi_t^b=\pi^a+\left(E_t\pi_{t+1}-\pi^a\right)/\delta$ in a Hybrid New-Keynesian Model

The Lagrangian expression of the problem equals

(I.1) $$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\left(\pi_t - \pi^a - \left(E_t \pi_{t+1} - \pi^a\right)/\delta\right)^2 + \lambda^b \left(x_t - x^a - \delta\left(x_{t-1} - x^a\right)\right)^2 \right] \\ + \psi_t \left[\kappa x_t + \phi \pi_{t-1} + (1-\phi)\beta \pi_{t+1} + u_t - \pi_t \right] \end{array} \right\}.$$

As the central bank operates with discretion, we only need to calculate the first-order conditions with respect to π_0, x_0, and x_1 as follows:

(I.2) $$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \pi_0} &= \pi_0 - \pi^a - \left(E_0 \pi_1 - \pi^a\right)/\delta - \psi_0 + \beta \phi \psi_1 = 0; \\ \frac{\partial \mathcal{L}}{\partial x_0} &= \lambda^b \left(x_0 - x^a - \delta\left(x_{-1} - x^a\right)\right) + \kappa \psi_0 - \beta \delta \lambda^b \left(E_0 x_1 - x^a - \delta\left(x_0 - x^a\right)\right) = 0; \text{ and} \\ \frac{\partial \mathcal{L}}{\partial x_1} &= \lambda^b \left(E_0 x_1 - x^a - \delta\left(x_0 - x^a\right)\right) + \kappa \psi_1 - \beta \delta \lambda^b \left(E_0 x_2 - x^a - \delta\left(x_1 - x^a\right)\right) = 0. \end{aligned}$$

Eliminating multipliers Ψ_0 and Ψ_1 gets

(I.3) $$\begin{aligned} &\kappa\left[\left(\pi_0 - \pi^a\right) - \left(E_0 \pi_1 - \pi^a\right)/\delta\right] \\ &\quad + \lambda^b \left\{ \left[\left(x_0 - x^a\right) - \delta\left(x_{-1} - x^a\right)\right] - \beta(\delta + \phi)\left[\left(E_0 x_1 - x^a\right) - \delta\left(x_0 - x^a\right)\right] \right\} \\ &\quad + \lambda^b \left\{ \phi \beta^2 \delta \left[\left(E_0 x_2 - x^a\right) - \delta\left(x_1 - x^a\right)\right] \right\} = 0. \end{aligned}$$

As we require that discretionary policy replicates the optimal policy, the optimal solutions in equations (4.2a) and (4.2b) from the timeless perspective must satisfy equation (I.3). Using equations (4.2a) and (4.2b) for t=0,1,2, $E_0 u_1 = \rho u_0$, and $E_0 u_2 = \rho^2 u_0$ leads to

(I.4a) $$\left(\pi_0 - \pi^a\right) - \left(E_0 \pi_1 - \pi^a\right)/\delta = -\left[\phi\left(1 - \beta \rho^2\right) - (1-\rho)\right] \frac{\rho}{(\delta \rho d - a)} u_0,$$

(I.4b) $$\left(x_0 - x^a\right) - \delta\left(x_{-1} - x^a\right) = -\frac{\kappa \delta \rho}{\lambda(\delta \rho d - a)} u_0,$$

(I.4c) $$\left(E_0 x_1 - x^a\right) - \delta\left(x_0 - x^a\right) = -\frac{\kappa \delta \rho}{\lambda(\delta \rho d - a)} \rho u_0, \text{ and}$$

(I.4d) $$\left(E_0 x_2 - x^a\right) - \delta\left(E_0 x_1 - x^a\right) = -\frac{\kappa\delta\rho}{\lambda\left(\delta\rho d - a\right)}\rho^2 u_0 .$$

Substituting equations (I.4a), (I.4b), (I.4c), and (I.4d) into equation (I.3) generates λ^b as follows:

(I.5) $$\lambda^b = \frac{\left(\phi_c - \phi\right)\left(1-\beta\rho^2\right)}{\delta\left(1-\delta\beta\rho\right)\left(1-\phi\beta\rho\right)}\lambda$$

where ϕ_c is denoted by (4.11).

Appendix J: Weight Response to Structural Parameters in a Backward-Looking Model ($\phi_c < \phi < 1$)

We begin with equation (4.10) as follows:

(J.1) $$\lambda^b = \frac{\left(\phi-\phi_c\right)\left(1-\beta\rho^2\right)}{\rho\left(1-\phi\beta\rho\right)}\lambda = \frac{\rho\left(1-\phi\beta\rho\right)-\left(1-\phi\right)}{\rho\left(1-\phi\beta\rho\right)}\lambda = \left[1-\frac{\left(1-\phi\right)}{\rho\left(1-\phi\beta\rho\right)}\right]\lambda < \lambda .$$

Differentiating λ^b with respect to ϕ leads to

(J.2) $$\frac{\partial\lambda^b}{\partial\phi} = \frac{\rho\left(1-\phi\beta\rho\right)-\left(1-\phi\right)\beta\rho^2}{\left[\rho\left(1-\phi\beta\rho\right)\right]^2}\lambda = \frac{\left(1-\beta\rho\right)}{\rho\left(1-\phi\beta\rho\right)^2}\lambda > 0.$$

Differentiating λ^b with respect to β generates

(J.3) $$\begin{aligned}\frac{\partial\lambda^b}{\partial\beta} &= \frac{-\phi\rho^2\rho\left(1-\phi\beta\rho\right)+\left[\rho\left(1-\phi\beta\rho\right)-\left(1-\phi\right)\right]\phi\rho^2}{\left[\rho\left(1-\phi\beta\rho\right)\right]^2} \\ &= \frac{-\left(1-\phi\right)\phi}{\left(1-\phi\beta\rho\right)^2} < 0\end{aligned} .$$

Differentiating λ^b with respect to λ leads to

(J.4) $$\frac{\partial\lambda^b}{\partial\lambda} = \frac{\left(\phi-\phi_c\right)\left(1-\beta\rho^2\right)}{\rho\left(1-\phi\beta\rho\right)} > 0 .$$

Differentiating λ^b with respect to κ results in

(J.5) $$\frac{\partial \lambda^b}{\partial \kappa} = 0 .$$

Differentiating λ^b with respect to ρ produces

(J.6) $$\begin{aligned}\frac{\partial \lambda^b}{\partial \rho} &= \frac{(1-2\phi\beta\rho)\rho(1-\phi\beta\rho)-[\rho(1-\phi\beta\rho)-(1-\phi)](1-2\phi\beta\rho)}{[\rho(1-\phi\beta\rho)]^2} \\ &= \frac{(1-\phi)(1-2\phi\beta\rho)}{[\rho(1-\phi\beta\rho)]^2}\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \text{ as } (1-2\phi\beta\rho)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0.\end{aligned}$$

Appendix K: When Does the Weight Response Equal λ in the Forward-Looking Model ($0 < \phi < \phi_c$)

We begin with equation (4.14) as follows:

(K.1) $$\lambda^b = \frac{(\phi_c - \phi)(1-\beta\rho^2)}{\delta(1-\delta\beta\rho)(1-\phi\beta\rho)}\lambda$$

Note that λ^b does depend on δ, which depends on model parameters.

We solve for the coefficient of λ equal to one as follows:

(K.2) $$\frac{(\phi_c - \phi_{\lambda^b=\lambda})(1-\beta\rho^2)}{\delta(1-\delta\beta\rho)(1-\phi_{\lambda^b=\lambda}\beta\rho)} = 1 .$$

We can show that the following result holds:

(K.3) $$\phi_{\lambda^b=\lambda} = \frac{(1-\rho)-\delta(1-\delta\beta\rho)}{[(1-\beta\rho^2)-\delta\beta\rho(1-\delta\beta\rho)]},$$

where $\phi_{\lambda^b=\lambda}$ equals the value of ϕ, where $\lambda^b = \lambda$. The parameter $\phi_{\lambda^b=\lambda}$ may take on positive or negative values as well as exceeding or falling short of $\phi_c = \frac{(1-\rho)}{(1-\beta\rho^2)}$. We cannot compare λ^b and λ.

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