Target Level and Variability Trade-offs☆

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Abstract: Kydland and Prescott (1977) observe that optimal policy proves time inconsistent because of rational expectations. This paper shows that the issue of time inconsistency also arises from fewer policy instruments than targets. Fewer instruments, as is well known, lead to the issue of target controllability (Tinbergen, 1963). Accordingly, this paper investigates these two long-standing policy issues -- target controllability and time inconsistency, using the hybrid new-Keynesian model (Clarida *et al.*, 1999). This paper finds that to address the time inconsistency issue, policy makers must first address target controllability, and that a proper target level trade-off solves target controllability, whereas a proper relative weight between target stabilizations achieves optimal target variability trade-off and solves time inconsistency. Other main results include the following. This paper derives a necessary and sufficient condition for joint asymptotic controllability of target values. The condition, which we call the long-run target level trade-off equation, is identical under commitment and under discretion. When the time-inconsistency problem does not exist, the central bank must make the same target variability trade-offs as society desires.

Key words: Target controllability; Time inconsistency; Optimal policy; Discretionary policy; Target level; Target variability; Trade-off

J.E.L. Classification: E52, E58

☆ This paper is a research result of the Humanities and Social Science Project “Theoretical and empirical research on China monetary policy rules in the open economy” (Grant No. 10YJA790238) supported by the Ministry of Education of China, as well as of the Key Project “Research on macro-prudential policy, monetary policy, and governance arrangements” (Grant No. 12AJY011) supported by the National Social Science Fund of China.

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1. Introduction

Kydland and Prescott (1977) observe that optimal policy proves time inconsistent because of rational expectations. This paper shows that the issue of time inconsistency also arises from fewer policy instruments than targets. Fewer instruments, as is well known, lead to the issue of target controllability (Tinbergen, 1963). Accordingly, this paper investigates these two long-standing policy issues, and argues that policy makers must address target controllability before addressing time inconsistency.

In considering the time inconsistency of optimal policy (e.g., Kydland and Prescott [1977]; and Calvo [1978]), researchers typically delegate a loss function to the central bank to address this issue, for instance, the conservative central banker of Rogoff (1985), the inflation contract of Walsh (1995), the employment contract of Chortareas and Miller (2003), the inflation target of Svensson (1997), the nominal income growth target of Beetsma and Jensen (1999) and Jensen (2002), the price-level target of Vestin (2006), the consistent target of Yuan, *et al.* (2011), and so on. Though the various delegation schemes ostensibly differ and lead to different interpretations, several delegations prove essentially identical.[1] Under the delegated loss function, the central bank operates monetary policy with discretion. As a result, monetary policy is time consistent. The main issue, however, is whether discretionary policy can approximate or even reproduce optimal policy. This paper's delegation scheme makes discretionary policy

1. Yuan, *et al.* (2011) compare their designed loss function with the loss functions in Svensson (1997) without employment persistence, Walsh (1995), and Chortareas and Miller (2003), and show that the four loss functions generate identical results with respect to the policy decision. Svensson (1997) also observes that the inflation-target-conservative loss function without employment persistence mimics the linear inflation contract in Walsh (1995).

reproduce optimal policy.

The issue of target controllability comes from an older tradition of policy issue (e.g., Kalman [1960]; Tinbergen [1963]; Preston [1974]; and Aoki [1975]). Brockett and Mesarović (1965) define three types of controllability (reproducibility): point, path, and asymptotic controllability.[2] *Point controllability* means that policy makers can achieve certain target values at a specified point in time. In practice, policy makers probably desire a stronger notion than point controllability, path controllability. *Path controllability* means that policy makers can make target variables follow some prescribed trajectories over a certain time interval (e.g., Aoki [1975]). Obviously, path controllability implies point controllability, whereas the converse generally does not hold. Path controllability plays a growing role in dynamic models of economic policy.[3] *Asymptotic controllability* means that policy makers can reach the target values at infinity. This paper considers both path and asymptotic controllability, which prove useful concepts in describing the equilibrium paths of target variables.

This paper addresses both issues with the delegation approach by designing the appropriate central bank loss function that determines the central bank's long- and short-run

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2 . Brockett and Mesarović (1965) use "reproducibility" instead of "controllability". The terminology "reproducibility" appears in the engineering literature, whereas the same concept of "controllability" appears in the economics literature. They introduce four rather than three types of controllability: point, locally path, uniformly path, and asymptotic controllability. We here refer to both locally and uniformly path controllability as path controllability. In addition, the literature also calls path controllability, functional or perfect controllability.

3. See, for example, Nyberg and Viotti (1978), Buiter and Gersovitz (1981, 1984), Wohltmann (1981, 1985), Tondini (1984), and Maas and Nijmeijer (1994). Besides extending Tinbergen's "static controllability" to dynamic controllability (Preston [1974]), the literature extends controllability from one-decision-maker to multiple-decision-makers (game) context. See Acocella and Di Bartolomeo (2006), Acocella et al. (2006, 2007), and Hughes Hallett et al. (2010, 2012) for controllability in a game context.

target values of inflation and output as well as the relative weight between stabilizing inflation and output. Intuitively, a proper target value trade-off solves the issue of target controllability, and a proper weight achieves optimal target stabilization trade-off and solves time inconsistency.

The delegated long- and short-run target values and weight parameter play roles as follows. The long-run target values guide monetary policy towards a correct end, resulting in asymptotic controllability and eliminating the constant average inflation bias.[4] Though the central bank, using discretion, moves toward a correct end, it still deviates from the optimal equilibrium paths during the evolution toward the correct end, causing a welfare loss. The short-run target values further bind monetary policy to follow the optimal equilibrium paths, resulting in path controllability and removing state-contingent inflation bias. The relative weight eliminates the stabilization bias, making the target stabilization trade-off with discretion under the central bank loss function replicate the target stabilization trade-off with commitment under the social loss function. As a result, the three biases disappear. Discretionary policy under the central bank loss function replicates optimal policy under the social loss function. Meanwhile, the central bank achieves the delegated target values, establishing monetary policy credibility.

The paper unfolds as follows. Sections 2 and 3 set the stage for our analysis in sections 4, 5, and 6. Section 2 determines both optimal and discretionary equilibrium paths of target variables in a hybrid new-Keynesian model, given a social loss function. Section 3 summarizes the results of Section 2 in a proposition, the necessary and sufficient condition for joint

4. See Svensson (1997) for the three biases - the constant average inflation bias, the state-contingent inflation bias, and the stabilization bias.

asymptotic controllability of target values with commitment and with discretion. Section 4 develops an approach to solve both target controllability and time inconsistency by designing the central bank loss function. Section 5 concretely designs central bank loss functions for the hybrid new-Keynesian model, using the approach in Section 4. Section 6 discusses the delegated weight to deepen our understanding of time inconsistency. Section 7 summarizes and concludes.

2. Optimal policy and discretionary policy

To develop our analysis, we simply adopt the hybrid new-Keynesian model in Clarida *et al.* (1999) without much description. The model combines the possibility of inflation expectations and inflation inertia, and reduces to a purely forward-looking or a purely backward-looking model by choosing the extreme values of the parameter that indexes expectations across the forward- and backward- looking dimensions. See Clarida *et al.* (1999) for more details.

The social intertemporal loss function equals

$$\mathbb{L} = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t\right), \tag{1}$$

where β ($0 < \beta < 1$) is the discount factor and E is the expectations operator. The period loss function equals the following:

$$L_t = \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\left(x_t - x^*\right)^2\right], \tag{2}$$

where π is the inflation rate, x is the output gap between the actual and natural output levels, π^* is the socially desirable inflation rate, x^* is the gap between the efficient and natural levels, and λ is the social weight on output stabilization relative to inflation stabilization around their respective target values. Generally, imperfections and/or distortions exist, and, thus, we assume that $x^* > 0$.

Aggregate supply equals an expectations-augmented Phillips curve with forward-looking expectations and endogenous inflation

$$\pi_t = \kappa x_t + \phi \pi_{t-1} + (1-\phi)\beta E_t \pi_{t+1} + u_t, \tag{3}$$

where κ (κ >0) is the sensitivity of the inflation rate to the output gap, ϕ indexes the degree of lagged versus expected future inflation rates, and u_t is a cost-push shock that follows an AR(1) process, $u_t = \rho u_{t-1} + \hat{u}_t$, where $0 \le \rho < 1$, and $\hat{u}_t$ is a white noise residual.

We do not introduce aggregate demand (IS curve), which involves a nominal interest rate as the policy instrument. Once we determine the optimal paths for $\{\pi_t, x_t\}_{t=0}^{\infty}$ using the social loss function and the Phillips curve, both of which do not involve the interest rate, then we can pin down the optimal path of interest rates through the IS curve. So the Phillips curve proves critical for policy.

2.1. Optimal policy, time inconsistency, and target controllability

2.1.1. Optimal policy

The consolidated first-order condition of optimal policy under the social intertemporal loss function (1) with period loss function (2) subject to the Phillips curve in Eq. (3) equals[5]

$$\left(x_0 - x^*\right) - \phi\beta\left(E_0 x_1 - x^*\right) = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right) \qquad \text{for } t = 0, \tag{4a}$$

$$\begin{aligned}\left(E_0 x_t - x^*\right) - \phi\beta\left(E_0 x_{t+1} - x^*\right) - (1-\phi)\left(E_0 x_{t-1} - x^*\right) \\ = -\frac{\kappa}{\lambda}\left(E_0 \pi_t - \pi^*\right)\end{aligned} \qquad \text{for } t \ge 1. \tag{4b}$$

Combining the first-order conditions (4a) and (4b) and the Phillips curve (Eq. 3) leads to optimal

5. See Eqs. (A.3a) and (A.3b) in Appendix A.

equilibrium paths of target variables[6]

$$\left(\pi_t-\hat{\pi}^{ss}\right)=\delta\left(\pi_{t-1}-\hat{\pi}^{ss}\right)+\lambda q\left[\left(1-\phi\beta\rho\right)u_t-\left(1-\phi\right)u_{t-1}\right] \quad \text{for } t\geq 1, \tag{5a}$$

$$\left(x_t-\hat{x}^{ss}\right)=\delta\left(x_{t-1}-\hat{x}^{ss}\right)-\kappa q u_t \quad \text{for } t\geq 1, \tag{5b}$$

where δ is a root of the characteristic equation

$$a\beta^2\delta^4-\beta\delta^3+b\delta^2-\delta+a=0 \quad \text{with} \tag{6a}$$

$$a\equiv\phi\left(1-\phi\right), \tag{6b}$$

$$b\equiv 1+\kappa^2/\lambda+\phi^2\beta+\left(1-\phi\right)^2\beta=1+\beta+\kappa^2/\lambda-2a\beta, \tag{6c}$$

$$q\equiv\frac{\delta\rho}{\lambda\left(\delta\rho d-a\right)}, \tag{6d}$$

$$d\equiv a\beta^2\left(\delta^2+\delta\rho+\rho^2\right)-\beta\left(\delta+\rho\right)+b, \tag{6e}$$

$$\hat{\pi}^{ss}\equiv\frac{\kappa}{\kappa^2+\lambda a\left(1-\beta\right)^2}\left[\kappa\pi^*+\lambda\phi\left(1-\beta\right)x^*\right], \text{ and} \tag{6f}$$

$$\hat{x}^{ss}\equiv\left[\left(1-\phi\right)\left(1-\beta\right)/\kappa\right]\hat{\pi}^{ss}. \tag{6g}$$

For the purely forward-looking, new-Keynesian model (ϕ = 0), the solution in Eqs. (5a) and (5b) reduces to the solution in Clarida *et al.* (1999, 1703-1704).[7]

We discuss two interesting parameters, which are important for designing the delegation mechanism. First, Eqs. (6a), (6b), and (6c) determine, what we call, the system convergence persistence, δ. We cannot determine whether a root $\delta\in(0,1)$ exists for any $\phi\in(0,1)$. But for

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6. See Eqs. (A.35) and (A.14) and their derivation in Appendix A. See (A.39) and (A.24) for the solution for t=0.

7. If ϕ = 0 or 1, then a = 0, $d=b-\beta\left(\delta+\rho\right)$, $d\delta=b\delta-\beta\delta^2-\delta\beta\rho$, and Eq. (6a) reduces to $\beta\delta^2-b\delta+1=0$. Thus, $b\delta-\beta\delta^2=1$. Therefore, $d\delta=1-\delta\beta\rho$ or $1/d=\delta/\left(1-\delta\beta\rho\right)$. Thus, $q=1/\left(\lambda d\right)=\delta/\left[\lambda\left(1-\delta\beta\rho\right)\right]$.

the two extreme cases of ϕ = 0 or 1, the characteristic Eq. (6a) reduces to $\beta\delta^2 - b\delta + 1 = 0$, and a root $\delta \in (0,1)$ does exist. That is, the system can converge under optimal policy for the two extreme cases. In addition, whether a root $\delta \in (0,1)$ exists, δ does not relate to the persistence of the cost-push shock, ρ. The irrelevance of the cost-push shocks reflects the quadratic loss function and the linear Phillips curve with additive shocks.

Second, observing the optimal equilibrium path of inflation rate, Eq. (5a), we find an interesting value. Using the AR(1) process of the cost-push shocks, transform Eq. (5a) into

$$\left(\pi_t - \hat{\pi}^{ss}\right) = \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right) + \lambda q\left[\left(\phi - \phi_c\right)\left(1 - \beta\rho^2\right)u_{t-1} + \left(1 - \phi\beta\rho\right)\hat{u}_t\right], \text{ where} \quad (7)$$

$$\phi_c \equiv \frac{1-\rho}{1-\beta\rho^2}, \text{ and } 0 < \phi_c \leq 1. \quad (8)$$

The critical value, ϕ_c, is meaningful. For convenience, we define the Phillips curve as principally backward-looking (forward-looking) if $\phi_c \leq \phi \leq 1$ ($0 \leq \phi \leq \phi_c$). Whether the Phillips curve exhibits principally backward- or forward-looking behavior results in completely different (positive or negative) responses of policy makers to cost-push shocks and, we will see in Section 5, very different delegations to the central bank.

The critical value ϕ_c depends on both the discount factor β and the cost-push shock persistence ρ. $\partial\phi_c/\partial\beta > 0$, and $\partial\phi_c/\partial\rho < 0$.[8] That is, a more important future and/or less persistent cost-push shocks lead to a higher critical value and, thus, the Phillips curve for a given inflation persistence ϕ more likely exhibits principally forward-looking behavior ($\phi \leq \phi_c$).

8. $\partial\phi_c/\partial\beta = \rho^2(1-\rho)\big/\left(1-\beta\rho^2\right)^2 > 0$ and $\partial\phi_c/\partial\rho = -\left[(1-\beta) + \beta(1-\rho)^2\right]\big/\left(1-\beta\rho^2\right)^2 < 0$ for $0 < \rho < 1$.

Accordingly, whether the Phillips curve exhibits principally forward- or backward-looking behavior depends on the three dynamic parameters, ϕ, β, and ρ, rather than merely ϕ.

2.1.2. Time inconsistency

The first-order conditions (4a) and (4b) suggest the time inconsistency of the optimal solution. Optimal policy requires that the present period follows condition (4a) and that future periods follow condition (4b). In practice, however, the central bank re-minimizes the loss function each period and, thus, always follows condition (4a).

2.1.3. Target controllability

In addition to time inconsistency, optimal policy still faces target controllability. We adopt optimality from the timeless perspective,[9] and analyze Eqs. (5a) and (5b) for $t \geq 1$. The optimal equilibrium paths of the target variables in Eqs. (5a) and (5b) consist of deterministic and random components. We denote the deterministic components as follows

$$\hat{\pi}_t \equiv \hat{\pi}^{ss} + \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right) = \delta\pi_{t-1} + \left(1-\delta\right)\hat{\pi}^{ss}, \tag{9a}$$

$$\hat{x}_t \equiv \hat{x}^{ss} + \delta\left(x_{t-1} - \hat{x}^{ss}\right) = \delta x_{t-1} + \left(1-\delta\right)\hat{x}^{ss}, \tag{9b}$$

and random components related to cost-push shocks as follows

$$\hat{\pi}_t^u \equiv \lambda q\left[\left(1-\phi\beta\rho\right)u_t - \left(1-\phi\right)u_{t-1}\right], \text{ and} \tag{10a}$$

$$\hat{x}_t^u \equiv -\kappa q u_t. \tag{10b}$$

The deterministic components converge to $\left(\hat{\pi}^{ss}, \hat{x}^{ss}\right)$, if δ<1,

$$\hat{\pi}^{ss} = \lim_{t\to\infty} \hat{\pi}_t, \text{ and} \tag{11a}$$

9. Woodford (1999) introduces the concept of optimality from a "timeless perspective," which means the policy the central bank "*to which it would have wished to commit itself to at a date far in the past.*" (293, italics in original).

$$\hat{x}^{ss} = \lim_{t\to\infty} \hat{x}_t . \tag{11b}$$

Thus, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$ are the steady-state values of the inflation rate and the output gap under optimal policy.

As Eqs. (9) show, the deterministic component of the inflation rate (the output gap) in each period, $\hat{\pi}_t\left(\hat{x}_t\right)$, equals a weighted average of its lag, $\pi_{t-1}\left(x_{t-1}\right)$, and its steady-state value, $\hat{\pi}^{ss}\left(\hat{x}^{ss}\right)$, with weight δ on its lag. In other words, the two target variables evolve to their respective steady-state values with the system convergence persistence, δ.

Target controllability does not depend on random shocks in our specific model with a quadratic loss function, a linear Phillips curve, and additive shocks. Accordingly, we ignore shocks and consider the deterministic components when we consider target controllability. The ideal social target values of inflation rate and output gap are π^* and x^* whereas the realized target values in the long run under optimal policy equal $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$. Thus, the policy makers generally cannot asymptotically control either target, since $\hat{\pi}^{ss} \neq \pi^*$ and $\hat{x}^{ss} \neq x^*$.[10]

2.1.4. Necessary and sufficient condition for joint asymptotic controllability

Though the social target values are generally not controllable, we obtain a necessary and sufficient condition for joint asymptotic controllability of both target values. That is, both target values, π^* and x^*, of the inflation rate and the output gap are jointly asymptotically controllable under timeless perspective optimal policy if and only if they satisfy the condition

$$\pi^* = \kappa x^* + \phi\pi^* + \left(1-\phi\right)\beta\pi^* . \tag{12}$$

Proof. Sufficiency. Rearranging Eq. (12) gives

10. If $\phi = 0$, then the policy makers can asymptotically control the inflation rate ($\hat{\pi}^{ss} = \pi^*$).

$$x^* = \left[(1-\phi)(1-\beta)/\kappa\right]\pi^* . \tag{13}$$

Substituting Eq. (13) into Eq. (6f) generates $\hat{\pi}^{ss} = \pi^*$. Using Eq. (6g), $\hat{\pi}^{ss} = \pi^*$, and Eq. (13) in sequence produces $\hat{x}^{ss} = x^*$. That is, the two target values, π^* and x^*, are jointly asymptotically controllable.

Necessity. Eq. (6g) equals the steady-state Phillips curve

$$\hat{\pi}^{ss} = \kappa\hat{x}^{ss} + \phi\hat{\pi}^{ss} + (1-\phi)\beta\hat{\pi}^{ss} . \tag{14}$$

If the two target values, π^* and x^*, are jointly asymptotically controllable, then $\hat{\pi}^{ss} = \pi^*$ and $\hat{x}^{ss} = x^*$. Replacing $\hat{\pi}^{ss}$ with π^* and $\hat{x}^{ss}$ with x^* in Eq. (14) produces Eq. (12). QED

The necessary and sufficient condition in Eq. (12) equals the steady-state Phillips curve in Eq. (14), and exhibits a trade-off between target values. Accordingly, we call Eq. (12) the long-run target level trade-off equation.

2.2. Discretionary policy and target controllability

2.2.1. Discretionary policy

The equilibrium paths of inflation rate and output gap under discretion equal[11]

$$\left(\pi_t - \bar{\pi}^{ss}\right) = \bar{\delta}\left(\pi_{t-1} - \bar{\pi}^{ss}\right) + \lambda\bar{q}\left(1-\phi\beta\rho\right)u_t , \text{ and} \tag{15a}$$

$$\left(x_t - \bar{x}^{ss}\right) = \bar{\delta}\left(x_{t-1} - \bar{x}^{ss}\right) - \kappa\bar{q}\left(u_t - \phi\beta\bar{\delta}\rho u_{t-1}\right)/\left(1-\phi\beta\bar{\delta}\right), \tag{15b}$$

where $\bar{\delta}$ is the root of the characteristic equation

$$a\beta^2\bar{\delta}^3 - \beta\bar{\delta}^2 + b'\bar{\delta} - \phi = 0 , \tag{16a}$$

$$b' \equiv 1 + \kappa^2/\lambda + \phi^2\beta , \tag{16b}$$

11. See Eqs. (B.18) and (B.10) and their derivation in Appendix B.

$$\bar{q} \equiv \frac{1}{\lambda\left\{\phi/\bar{\delta} - \beta\rho\left[1 - a\beta\left(\bar{\delta} + \rho\right)\right]\right\}}, \tag{16c}$$

$$\bar{\pi}^{ss} \equiv \frac{\kappa}{\kappa^2 + \lambda(1-\phi)(1-\beta)(1-\phi\beta)}\left[\kappa\pi^* + \lambda(1-\phi\beta)x^*\right], \text{ and} \tag{16d}$$

$$\bar{x}^{ss} \equiv \left[(1-\phi)(1-\beta)/\kappa\right]\bar{\pi}^{ss}. \tag{16e}$$

Does a less-than-one root exist for the characteristic equation (16a)? Consider the two extreme cases of $\phi = 0$ and $\phi = 1$. For the case $\phi = 0$, the characteristic equation (16a) reduces to $\beta\bar{\delta}^2 - \left(1+\kappa^2/\lambda\right)\bar{\delta} = 0$, and its two roots are $\bar{\delta} = 0$, which is a degenerative solution, and $\bar{\delta} = \left(1+\kappa^2/\lambda\right)/\beta > 1$. The system diverges under the discretionary policy when $\phi = 0$, in contrast to converging under the optimal policy. For the case $\phi = 1$, the characteristic equation (16a) reduces to $\beta\bar{\delta}^2 - \left(1+\kappa^2/\lambda+\beta\right)\bar{\delta} + 1 = 0$. The larger root exceeds one and the smaller root lies between zero and one. The system can converge under discretion when $\phi = 1$. When the lag index ϕ goes from 0 to 1, the system changes from divergence to convergence. The system may also go through divergence, convergence, divergence, convergence, and so on.

2.2.2. Target controllability

Similar to optimal policy, the discretionary equilibrium paths of the target variables in Eqs. (15) consist of deterministic and random components. We denote the deterministic components as follows

$$\bar{\pi}_t \equiv \bar{\pi}^{ss} + \bar{\delta}\left(\pi_{t-1} - \bar{\pi}^{ss}\right) = \bar{\delta}\pi_{t-1} + \left(1-\bar{\delta}\right)\bar{\pi}^{ss}, \tag{17a}$$

$$\bar{x}_t \equiv \bar{x}^{ss} + \bar{\delta}\left(x_{t-1} - \bar{x}^{ss}\right) = \bar{\delta}x_{t-1} + \left(1-\bar{\delta}\right)\bar{x}^{ss}, \tag{17b}$$

and random components or reactions to cost-push shocks as follows

$$\bar{\pi}_t^u \equiv \lambda\bar{q}\left(1-\phi\beta\rho\right)u_t, \text{ and} \tag{18a}$$

$$\bar{x}_t^u \equiv -\kappa \bar{q}\left(u_t - \phi\beta\bar{\delta}\rho u_{t-1}\right) / \left(1 - \phi\beta\bar{\delta}\right). \tag{18b}$$

The deterministic components converge to $\left(\bar{\pi}^{ss}, \bar{x}^{ss}\right)$, if $\bar{\delta} < 1$,

$$\bar{\pi}^{ss} = \lim_{t\to\infty} \bar{\pi}_t \text{, and} \tag{19a}$$

$$\bar{x}^{ss} = \lim_{t\to\infty} \bar{x}_t . \tag{19b}$$

Thus, $\bar{\pi}^{ss}$ and $\bar{x}^{ss}$ are the steady-state values of the inflation rate and the output gap under discretionary policy.

The ideal social target values of inflation rate and output gap are π^* and x^* whereas the steady-state values equal $\bar{\pi}^{ss}$ and $\bar{x}^{ss}$. Similar to optimal policy, the policy maker, using discretion, generally cannot asymptotically control either inflation rate or output gap ($\bar{\pi}^{ss} \neq \pi^*$ and $\bar{x}^{ss} \neq x^*$).

2.2.3. Necessary and sufficient condition for joint asymptotic controllability

The same conclusion emerges under discretionary policy. Both target values, π^* and x^*, of the inflation rate and the output gap are jointly asymptotically controllable under discretionary policy if and only if they satisfy the condition in Eq. (12).

Proof. Sufficiency. If π^* and x^* satisfy Eq. (12), then Eq. (13) holds. Substituting Eq. (13) into Eq. (16d) generates $\bar{\pi}^{ss} = \pi^*$. Using Eq. (16e), $\bar{\pi}^{ss} = \pi^*$, and Eq. (13) in sequence produces $\bar{x}^{ss} = x^*$. That is, the two target values, π^* and x^*, are asymptotically controllable.

Necessity. Eq. (16e) equals the steady-state Phillips curve

$$\bar{\pi}^{ss} = \kappa\bar{x}^{ss} + \phi\bar{\pi}^{ss} + \left(1-\phi\right)\beta\bar{\pi}^{ss} . \tag{20}$$

If the two target values, π^* and x^*, are jointly asymptotically controllable, then $\bar{\pi}^{ss} = \pi^*$ and $\bar{x}^{ss} = x^*$. Replacing $\bar{\pi}^{ss}$ with π^* and $\bar{x}^{ss}$ with x^* in Eq. (20) produces Eq. (12). QED

3. Target level and variability trade-offs

We summarize the necessary and sufficient condition for joint asymptotic controllability of target values in a proposition. We also show the regressive nature of steady-state values of target variables, and indicate that the constant average inflation bias comes from fewer instruments than targets, rather than from time inconsistency as the literature stresses. In addition, target level and target variability trade-offs correspond respectively to target controllability and time inconsistency, and we argue that policy makers must address target controllability before addressing time inconsistency.

3.1. Necessary and sufficient condition for joint asymptotic controllability

The controllability literature usually considers the equations of motion, the available instruments, and the initial state of the system. Rather, we find the necessary and sufficient condition for the joint asymptotic controllability of target values.

Proposition. *If the system, with period social loss function in Eq. (2) subject to a linear Phillips curve in Eq. (3), converges with commitment,* π^* *and* x^* *are jointly asymptotically controllable if and only if they satisfy the steady-state Phillips curve in Eq. (12). The result also holds with discretion.*

Proof. We prove the proposition for optimal policy in a more simple way.

Sufficiency. If π^* and x^* satisfy Eq. (12), then combining Eq. (12) with the Phillips curve produces the Phillips curve around the target values as follows

$$\left(\pi_t-\pi^*\right)=\kappa\left(x_t-x^*\right)+\phi\left(\pi_{t-1}-\pi^*\right)+\left(1-\phi\right)\beta\left(E_t\pi_{t+1}-\pi^*\right)+u_t\,. \qquad (21)$$

Now, the optimization problem equals the minimization of the social intertemporal loss function with period loss function (2) subject to the Phillips curve around the target values in Eq. (21). Obviously, the equilibrium paths $\left(\pi_t - \pi^*, x_t - x^*\right)$ under optimal policy will converge to (0, 0), if the system converges. That is, $\left(\pi_t, x_t\right)$ converge to $\left(\pi^*, x^*\right)$. Therefore, π^* and x^* are jointly asymptotically controllable.

Necessity. If π^* and x^* are jointly asymptotically controllable under optimal policy, then $\hat{\pi}^{ss} = \pi^*$ and $\hat{x}^{ss} = x^*$ hold. The steady-state Phillips curve under optimal policy in Eq. (14) leads to the condition in Eq. (12).

The proof for discretionary policy follows exactly the same as that for optimal policy by using Eq. (20) instead of Eq. (14) and by noting that the difference between discretion and commitment merely reflects the different sequence of optimizations of decision makers. QED

3.2. Target level trade-off

3.2.1. The Tinbergen rule ($\lambda = 0$)

Let λ=0, and an equal number of instruments and targets exists -- one instrument and one target. In this case, the central bank only cares about inflation. It always sets and achieves $\pi_t = \pi^*$ whereas output fluctuates with shocks, $x_t = (1-\phi)(1-\beta)\pi^*/\kappa - u_t/\kappa$. Time inconsistency does not exist.

3.2.2. Target level trade-off ($\lambda \neq 0$)

If $\lambda \neq 0$, fewer instruments than the number of targets exist -- one instrument and two targets. In this case, target values are generally uncontrollable and time inconsistency exists as we discuss in Section 2. The proposition suggests that if the social target values happen to satisfy the steady-state Phillips curve, they are jointly asymptotically controllable. Joint asymptotic controllability reflects the divine coincidence (Blanchard and Galí [2007]).[12]

Obviously, infinite pairs of target values can lead to joint asymptotic controllability. Consider the pair, $\pi^* = 0$ and $x^* = 0$. That is, a zero inflation rate corresponds to the efficient equilibrium $(x^* = 0)$, and vice versa. Usually, monopolistic and/or tax distortions exist, resulting in the natural output below the efficient output $(x^* > 0)$. The inefficiency requires a positive inflation rate $(\pi^* > 0)$ according to Eq. (13). This explains to some extent why targeting a positive inflation rate prevails in the real world.

3.2.3. The regressive nature of steady-state values of target variables

The social target values, π^* and x^*, which come from the households utility function, generally do not satisfy the steady-state Phillips curve, which reflects the behavior of the firm. Therefore, the target values, π^* and x^*, are generally uncontrollable under both commitment and discretion. The steady-state values of target variables, however, exhibit regression to the target values, π^* and x^*. To see this, transform the expressions of steady-state values under commitment and under discretion in Eqs. (6f), (6g), (16d) and (16e) as follows:

$$\hat{\pi}^{ss} = \pi^* - \frac{\lambda\phi(1-\beta)}{\kappa^2 + \lambda a(1-\beta)^2}\left[(1-\phi)(1-\beta)\pi^* - \kappa x^*\right], \tag{22a}$$

$$\hat{x}^{ss} = x^* - \frac{\kappa}{\kappa^2 + \lambda a(1-\beta)^2}\left[\kappa x^* - (1-\phi)(1-\beta)\pi^*\right], \tag{22b}$$

$$\bar{\pi}^{ss} = \pi^* - \frac{\lambda(1-\phi\beta)}{\kappa^2 + \lambda(1-\phi)(1-\beta)(1-\phi\beta)}\left[(1-\phi)(1-\beta)\pi^* - \kappa x^*\right], \text{ and} \tag{23a}$$

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12. Blanchard and Galí (2007) define "divine coincidence" to mean that the stabilization of the inflation rate automatically leads to the stabilization of the output gap, joint stabilization. We borrow the concept here to mean joint asymptotic controllability, focusing on the levels rather than the variability of inflation and output.

$$\bar{x}^{ss} = x^* - \frac{\kappa}{\kappa^2 + \lambda(1-\phi)(1-\beta)(1-\phi\beta)}\left[\kappa x^* - (1-\phi)(1-\beta)\pi^*\right]. \tag{23b}$$

The expressions in the brackets equal the form of the steady-state Phillips curve, indicating that target values, π^* and x^*, are achievable when they satisfy the steady-state Phillips curve. The negative coefficients before the brackets reveal the regressive process. With a high target value of output gap (relative to that of inflation rate, i.e., $\kappa x^* > (1-\phi)(1-\beta)\pi^*$), the steady-state value of the output gap falls below its target value (i.e., $\hat{x}^{ss} < x^*$ and $\bar{x}^{ss} < x^*$), whereas the steady-state value of the inflation rate rises above its target value (i.e., $\hat{\pi}^{ss} > \pi^*$ and $\bar{\pi}^{ss} > \pi^*$).

The regressive nature of the steady-state values of target variables explains the constant average inflation bias. An overly ambitious output target value implies a relatively low inflation target value. A low inflation target value leads to the steady-state value of inflation higher than its target value, resulting in the constant average inflation bias. The constant average inflation bias, however, disappears when target values satisfy the steady-state Phillips curve. Therefore, the constant average inflation bias comes from fewer instruments than targets.

Both the literature and we agree that the inflation bias results from an overly ambitious output target value. The reason for the inflation bias, however, is different. The literature stresses that the inflation bias arises from time inconsistency, whereas we argue that the constant average inflation bias comes from target controllability and agree that the state-contingent inflation bias and the stabilization bias do arise from time inconsistency.

3.3. Target variability trade-off

The proposition suggests that a trade-off exists between target values. A trade-off also exists between target stabilization, as Eqs. (10) and (18) show. Obviously, the target stabilization trade-off under commitment differs from that under discretion, resulting in the stabilization bias, which prevails in discretionary policy. Target level and target variability trade-offs correspond respectively to target controllability and time inconsistency.

In summary, target level trade-off occurs in the long run, independent of policy implementation style -- commitment or discretion, as the proposition reveals. The target stabilization trade-off occurs in the short run and depends on the policy implementation style. In addition, the observation that the constant average inflation bias comes from target controllability, rather than from time inconsistency as the literature stresses, implies that the literature mistakes target controllability for time inconsistency. As a result, we address target controllability before addressing time inconsistency, avoiding entangling the two issues when addressing time inconsistency.

4. An approach to solving both target controllability and time inconsistency

We develop an approach to solving both target controllability and time inconsistency. Intuitively, a proper target value trade-off solves target controllability, whereas a proper relative weight between target stabilizations achieves the optimal target variability trade-off and solves time inconsistency. As a result, we design the central bank loss function by determining the central bank's long- and short-run target values of the inflation rate and the output gap, as well as the weight on stabilizing the output gap relative to stabilizing the inflation rate, given the social loss function and economic structures. The long- and short-run target values and the weight parameter play important roles in the delegation mechanism, as stated in the introduction. Assume that the central bank operates policy discretionarily.

4.1. Joint asymptotic controllability with long-run target values

We denote π^b, x^b and λ^b , respectively, the long-run target values of the inflation rate, the output gap, and the relative weight on stabilizing output relative to stabilizing inflation. That is, the central bank period loss function equals

$$L_t^b = \frac{1}{2}\left[\left(\pi_t - \pi^b\right)^2 + \lambda^b\left(x_t - x^b\right)^2\right]. \tag{24}$$

Now, we determine the long-run target values π^b and x^b. As we know, the social target values, π^* and x^*, are generally uncontrollable, and what we can best achieve in the long run is $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$ in Eqs. (6f) and (6g), the steady-state values under commitment. They completely depend on the model structural parameters. With a pragmatic attitude, we should delegate the moderate target values of $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$, not the ambitious values π^* and x^*, to the central bank. That is,

$$L_t^b = \frac{1}{2}\left[\left(\pi_t - \hat{\pi}^{ss}\right)^2 + \lambda^b\left(x_t - \hat{x}^{ss}\right)^2\right]. \tag{25}$$

Since $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$ satisfy the steady-state Phillips curve, they are jointly asymptotically controllable with discretion according to the proposition. That is,

$$\left(\tilde{\pi}^{ss}, \tilde{x}^{ss}\right) = \left(\hat{\pi}^{ss}, \hat{x}^{ss}\right) = \left(\pi^b, x^b\right), \tag{26}$$

where $\tilde{\pi}^{ss}$ and $\tilde{x}^{ss}$ denote, respectively, the steady-state values of the inflation rate and the output gap with discretion under the central bank loss function. Actually, we can control $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$ by noting the loss function (25) and the Phillips curve around the optimal steady state, using Eqs. (3) and (14),

$$\left(\pi_t - \hat{\pi}^{ss}\right) = \kappa\left(x_t - \hat{x}^{ss}\right) + \phi\left(\pi_{t-1} - \hat{\pi}^{ss}\right) + (1-\phi)\beta\left(E_t\pi_{t+1} - \hat{\pi}^{ss}\right) + u_t, \tag{27}$$

which is equivalent to the original Phillips curve.

Moderate and achievable target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$, establish monetary policy credibility.

4.2. Joint path controllability with short-run target values

The desired trajectories of the target variables equal the optimal equilibrium paths, $\{\hat{\pi}_t, \hat{x}_t\}_{t=0}^{\infty}$, not merely the end points $\left(\hat{\pi}^{ss}, \hat{x}^{ss}\right)$. To achieve path controllability (to bind the central bank to follow the optimal equilibrium paths), assume a state-contingent, short-run target values, denoted as π_t^b and x_t^b. Three requirements must exist for meaningful short-run target values. One, $\lim_{t\to\infty}\pi_t^b = \pi^b$ and $\lim_{t\to\infty}x_t^b = x^b$. That is, we achieve the long-run target values step by step through short-run target values. Two, the short-run target values must evolve with persistence δ to ensure that target variables also evolve, on average, with persistence δ. Three, we must impose predetermined short-run target values. The predetermined target values, though state-contingent, are feasible in practice. We modify the central bank loss function in Eq. (24) as follows:

$$L_t^b = \frac{1}{2}\left[\left(\pi_t - \pi_t^b\right)^2 + \lambda^b\left(x_t - x_t^b\right)^2\right]. \tag{28}$$

We specify π_t^b and x_t^b so that under the loss function (28), the deterministic components of the equilibrium paths of the inflation rate and the output gap under discretion, denoted as $\{\tilde{\pi}_t, \tilde{x}_t\}_{t=0}^{\infty}$, exactly replicate the desired trajectories, $\{\hat{\pi}_t, \hat{x}_t\}_{t=0}^{\infty}$. That is,

$$\left(\tilde{\pi}_t, \tilde{x}_t\right) = \left(\hat{\pi}_t, \hat{x}_t\right) \quad \text{for all } t. \tag{29}$$

With the short-run target values, discretionary policy is path controllable and, thus, eliminates the constant average and state-contingent inflation biases. This leaves the stabilization bias.

4.3. Eliminating stabilization bias with a proper weight

We can eliminate this bias by determining a proper weight, λ^b, such that

$$\left(\tilde{\pi}_t^u, \tilde{x}_t^u\right) = \left(\hat{\pi}_t^u, \hat{x}_t^u\right), \tag{30}$$

where $\left(\tilde{\pi}_t^u, \tilde{x}_t^u\right)$ denote the reactions to shocks with discretion under the designed loss function. Actually, we pin down the weight through one of the two equations, $\tilde{\pi}_t^u = \hat{\pi}_t^u$ or $\tilde{x}_t^u = \hat{x}_t^u$. Once one equation holds, the other equation also holds because the Phillips curve links the inflation rate and the output gap.

Now, we design central bank loss functions using the above approach.

5. Designing the central bank loss function

5.1. Joint asymptotic controllability with long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$

The equilibrium paths of target variables with discretion under period loss function (25) equal[13]

$$\left(\pi_t - \hat{\pi}^{ss}\right) = \tilde{\delta}\left(\pi_{t-1} - \hat{\pi}^{ss}\right) + \lambda\tilde{q}\left(1-\phi\beta\rho\right)u_t, \text{ and} \tag{31a}$$

$$\left(x_t - \hat{x}^{ss}\right) = \tilde{\delta}\left(x_{t-1} - \hat{x}^{ss}\right) - \kappa\tilde{q}\left(u_t - \phi\beta\tilde{\delta}\rho u_{t-1}\right)\Big/\left(1-\phi\beta\tilde{\delta}\right), \tag{31b}$$

where $\tilde{\delta}$ is a root of the characteristic equation

$$a\beta^2\tilde{\delta}^3 - \beta\tilde{\delta}^2 + \tilde{b}'\tilde{\delta} - \phi = 0, \tag{32a}$$

$$\tilde{b}' \equiv 1 + \kappa^2/\lambda^b + \phi^2\beta, \text{ and} \tag{32b}$$

$$\tilde{q} \equiv \frac{1}{\lambda^b\left\{\phi/\tilde{\delta} - \beta\rho\left[1 - a\beta\left(\tilde{\delta}+\rho\right)\right]\right\}}. \tag{32c}$$

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13. See Eqs. (C.1) and (C.2) in Appendix C. The forms of the solution are exactly the same as those for discretionary policy with social target values, π^* and x^*, in Eqs. (15) and (16).

For the case $0 < \phi \leq 1$,[14] we can conclude that $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$ are jointly asymptotically controllable, if the characteristic equation (32a) possesses a root less than one, $\tilde{\delta} < 1$.

Generally, the system is not path controllable under the discretionary policy since $\tilde{\delta} \neq \delta$, where $\tilde{\delta}$ and δ are determined, respectively, by Eqs. (32a) and (6a). With the change of the inflation inertia index ϕ, the Phillips curve and the system exhibit different behavior. We discuss the delegations of the short-run target values and the relative weight based on the parameter ϕ.

5.2. Joint path controllability and elimination of stabilization bias when $\phi = 1$

The case $\phi = 1$ proves interesting. Specifically, if the Phillips curve in Eq. (3) exhibits purely backward-looking behavior with no expectations as follows:

$$\pi_t = \kappa x_t + \pi_{t-1} + u_t \quad \text{or} \quad \left(\pi_t - \hat{\pi}^{ss}\right) = \kappa\left(x_t - \hat{x}^{ss}\right) + \left(\pi_{t-1} - \hat{\pi}^{ss}\right) + u_t, \tag{33}$$

then we do not need to delegate short-run target values or to change the relative weight. We only need to delegate moderate long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$. That is, the central bank adopts the simple loss function as follows:

$$L_t^b = \frac{1}{2}\left[\left(\pi_t - \hat{\pi}^{ss}\right)^2 + \lambda\left(x_t - \hat{x}^{ss}\right)^2\right]. \tag{34}$$

With the simple loss function (Eq. 34), the discretionary policy proves path controllable and eliminates stabilization bias! That is, $\tilde{\delta} = \delta$ (optimal system convergence persistence), $\left(\tilde{\pi}_t, \tilde{x}_t\right) = \left(\hat{\pi}_t, \hat{x}_t\right)$ (path controllability), and $\tilde{\pi}_t^u = \hat{\pi}_t^u$ (optimal stabilization trade-off). [15]

Intuitively, when the Phillips curve equals Eq. (33), which does not involve expectations, the optimal problem reduces to a control theory, rather than game theory, problem. That is, only

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14. When $\phi = 0$, the system diverges. See Subsection 2.2.1.

15. Appendix C provides the details of the derivations.

one decision maker exists rather than multiple decision makers. Time inconsistency does not exist. The consolidated first-order conditions under optimal policy also reveal non-existence of time inconsistency. The condition for $t = 0$ (Eq. 4a) equals the condition for $t \geq 1$ (Eq. 4b), when $\phi = 1$, indicating that the optimal policy is time-consistent. Though time inconsistency does not exist, the issue of target controllability does under the social loss function. The target variables' steady-state values in Eqs. (6f) and (6g) with $\phi = 1$ equal $\left(\hat{\pi}^{ss}, \hat{x}^{ss}\right) = \left(\pi^* + (\lambda/\kappa)(1-\beta)x^*, 0\right)$. Thus, $\left(\hat{\pi}^{ss}, \hat{x}^{ss}\right) \neq \left(\pi^*, x^*\right)$, if $x^* \neq 0$. That is, the social target values, π^* and x^*, are uncontrollable, if $x^* \neq 0$.

In sum, when $\phi = 1$, time inconsistency does not exist and we only face target controllability. As a result, we do not need to delegate to the central bank the short-run target values, which commit or bind central bank behavior. We also do not need to delegate to the central bank a different weight from that of society to change the trade-off between target stabilizations. We solve target controllability by delegating moderate and achievable long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$, to the central bank.

5.3. Joint path controllability and elimination of stabilization bias when $\phi_c \leq \phi \leq 1$

5.3.1. Joint path controllability with lagged, short-run target values

Whether time inconsistency exists, as well as whether the system converges with the long-run target values under the discretionary policy, we consider delegating short-run target values. The requirements for short-run target values suggest that we can assume the following values

$$\pi_t^b = \hat{\pi}_t = \hat{\pi}^{ss} + \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right) = \delta\pi_{t-1} + (1-\delta)\hat{\pi}^{ss}, \text{ and} \tag{35a}$$

$$x_t^b = \hat{x}_t = \hat{x}^{ss} + \delta\left(x_{t-1} - \hat{x}^{ss}\right) = \delta x_{t-1} + (1-\delta)\hat{x}^{ss}. \tag{35b}$$

Obviously, the short-run target value, $\pi_t^b\left(x_t^b\right)$, converges to its long-run target value, $\hat{\pi}^{ss}\left(\hat{x}^{ss}\right)$, with persistence δ, and is predetermined. We modify the corresponding central bank period loss function in Eq. (25) as follows:

$$L_t^b = \frac{1}{2}\left\{\left[\left(\pi_t - \hat{\pi}^{ss}\right) - \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right)\right]^2 + \lambda^b\left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right]^2\right\}. \qquad (36)$$

The consolidated first-order condition of the problem under discretion equals[16]

$$\kappa\left\{\begin{array}{l}\left[\left(\pi_t - \hat{\pi}^{ss}\right) - \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right)\right] \\ -\beta\delta\left[\left(E_t\pi_{t+1} - \hat{\pi}^{ss}\right) - \delta\left(\pi_t - \hat{\pi}^{ss}\right)\right]\end{array}\right\} + \lambda^b\left\{\begin{array}{l}\left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right] \\ -\beta\left(\phi + \delta\right)\left[\left(E_t x_{t+1} - \hat{x}^{ss}\right) - \delta\left(x_t - \hat{x}^{ss}\right)\right] \\ +\phi\beta^2\delta\left[\left(E_t x_{t+2} - \hat{x}^{ss}\right) - \delta\left(E_t x_{t+1} - \hat{x}^{ss}\right)\right]\end{array}\right\} = 0. \qquad (37)$$

Combining the consolidated first-order condition in Eq. (37) and the Phillips curve around the optimal steady state in Eq. (27) produces the solution. Because shocks do not affect controllability, set $u_t = 0$. If $u_t = 0$, obviously, a solution equals

$$\left(\pi_t - \hat{\pi}^{ss}\right) = \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right), \text{ and} \qquad (38a)$$

$$\left(x_t - \hat{x}^{ss}\right) = \delta\left(x_{t-1} - \hat{x}^{ss}\right). \qquad (38b)$$

That is, there exists a discretionary policy, which proves path controllable, and, thus, eliminates the constant average and state-contingent inflation biases.

5.3.2. Elimination of stabilization bias with a proper weight

If $u_t \neq 0$, the stabilization bias arises. To eliminate this bias, $\tilde{\pi}_t^u = \hat{\pi}_t^u$ and $\tilde{x}_t^u = \hat{x}_t^u$ must hold. That is, using Eqs. (5a) and (5b) in Eq. (37) determines λ^b as follows:

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16. See Eq. (D.2) in Appendix D.

$$\frac{\lambda^b}{\lambda}(1-\delta\beta\rho)(1-\phi\beta\rho)u_t = \left[\phi\left(1-\beta\rho^2\right)-(1-\rho)\right](1-\delta\beta\rho)u_{t-1} + \left\{(1-\phi\beta\rho)-\delta\beta\left[\phi\left(1-\beta\rho^2\right)-(1-\rho)\right]\right\}\hat{u}_t, \tag{39}$$

where $\hat{u}_t$ is white noise. Policy makers cannot manage $\hat{u}_t$ even under commitment. Actually, we obtain the optimal solution under commitment by minimizing the intertemporal loss, forming rational expectations at the beginning, E_0. Now, applying E_{t-1} to Eq. (39) produces

$$\lambda^b = (\phi-\phi_c)\frac{\left(1-\beta\rho^2\right)}{\rho(1-\phi\beta\rho)}\lambda, \tag{40}$$

where ϕ_c is defined in Eq. (8). Though the discretionary solution slightly differs from the optimal solution, the expectations of the intertemporal losses equal each other under the two solutions.

We see that $\lambda^b = \lambda$ when $\phi = 1$. That is, when $\phi = 1$, either the long-run or the short-run target values can achieve path controllability, and the weight always equals the social weight.

Economical feasibility requires $\lambda^b \geq 0$. Thus, $\phi \geq \phi_c$.

5.4. Joint path controllability and elimination of stabilization bias when $0 \leq \phi \leq \phi_c$

5.4.1. Joint path controllability with expected, short-run target values

For the case $\phi \geq \phi_c$, the inflation rate in the Phillips curve (Eq.3) exhibits principally backward-looking behavior (π_{t-1}), and the short-run target value for the inflation rate in Eq. (35a) exhibits a lag. Now, for the case $\phi \leq \phi_c$, we guess that the short-run target value for the inflation rate in the principally forward-looking Phillips curve must incorporate expected inflation, $E_t\pi_{t+1}$. As a result, we construct the short-run target values as follows

$$\pi_t^b = \hat{\pi}^{ss} + \delta^{-1}\left(E_t\pi_{t+1} - \hat{\pi}^{ss}\right) = \delta^{-1}E_t\pi_{t+1} + \left(1-\delta^{-1}\right)\hat{\pi}^{ss}, \text{ and} \tag{41a}$$

$$x_t^b = \hat{x}^{ss} + \delta\left(x_{t-1} - \hat{x}^{ss}\right) = \delta x_{t-1} + (1-\delta)\hat{x}^{ss}. \tag{41b}$$

Obviously, the short-run target value, $\pi_t^b\left(x_t^b\right)$, converges to its long-run target value, $\hat{\pi}^{ss}\left(\hat{x}^{ss}\right)$, with persistence δ, and is predetermined.

The short-run inflation target value equals the 'weighted' ($\delta^{-1}>1$) average of the expected inflation rate and the long-run target value. Using an expectation as a target value seems somewhat strange, but we argue that its rationality depends on the forward-looking Phillips curve. Also, the expected inflation target value demonstrates the idea of "implementing optimal policy through inflation-forecast targeting" (Svensson and Woodford [2005]). We modify the corresponding central bank period loss function in Eq. (25) as follows:

$$L_t^b=\frac{1}{2}\left\{\left[\left(\pi_t-\hat{\pi}^{ss}\right)-\delta^{-1}\left(E_t\pi_{t+1}-\hat{\pi}^{ss}\right)\right]^2+\lambda^b\left[\left(x_t-\hat{x}^{ss}\right)-\delta\left(x_{t-1}-\hat{x}^{ss}\right)\right]^2\right\}. \quad (42)$$

The consolidated first-order condition of the problem under discretion equals[17]

$$\kappa\begin{bmatrix}\left(E_t\pi_{t+1}-\hat{\pi}^{ss}\right)\\ -\delta\left(\pi_t-\hat{\pi}^{ss}\right)\end{bmatrix}=\lambda^b\delta\begin{Bmatrix}\left[\left(x_t-\hat{x}^{ss}\right)-\delta\left(x_{t-1}-\hat{x}^{ss}\right)\right]\\ -\beta\left(\delta+\phi\right)\left[\left(E_tx_{t+1}-\hat{x}^{ss}\right)-\delta\left(x_t-\hat{x}^{ss}\right)\right]\\ +\phi\delta\beta^2\left[\left(E_tx_{t+2}-\hat{x}^{ss}\right)-\delta\left(E_tx_{t+1}-\hat{x}^{ss}\right)\right]\end{Bmatrix}. \quad (43)$$

Combining the consolidated first-order condition in Eq. (43) and the Phillips curve around the optimal steady state in Eq. (27) produces the solution. Once again, if $u_t=0$, a solution equals Eqs. (38). That is, there exists a discretionary policy, which proves path controllable, resulting in moving the constant average and state-contingent inflation biases.

5.4.2. Elimination of stabilization bias with a proper weight

If $u_t\neq 0$, the stabilization bias emerges. To eliminate this bias, $\tilde{\pi}_t^u=\hat{\pi}_t^u$ and $\tilde{x}_t^u=\hat{x}_t^u$ must hold. That is, using Eqs. (5a) and (5b) in Eq. (43) determines λ^b as follows

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17. See Eq. (E.2) in Appendix E.

$$\lambda^b = \left(\phi_c - \phi\right) \frac{\left(1-\beta\rho^2\right)}{\delta\left(1-\delta\beta\rho\right)\left(1-\phi\beta\rho\right)} \lambda. \tag{44}$$

Economical feasibility requires that $\lambda^b \geq 0$. Thus, $\phi \leq \phi_c$. For the case $0 \leq \phi \leq \phi_c$, the discretionary solution exactly replicates the optimal solution.

In sum, the short-run inflation target value conforms to the macroeconomic structure (i.e., Phillips curve). That is, it is lagged (expected), if the inflation rate in the Phillips curve exhibits principally backward-looking (forward-looking) behavior. We discuss the delegated weight and its interpretation in the next section. In short, with the delegated loss function, a discretionary policy proves path controllable (removing the constant average and state-contingent inflation biases), and eliminates stabilization bias, resulting in optimality.

6. Discussion on the delegated weight

The proposition solves target controllability. We discuss the delegated weight to deepen our understanding of time inconsistency.

6.1. The delegated weight for the purely backward-looking Phillips curve (ϕ = 1)

When the Phillips curve exhibits purely backward-looking behavior with no expectations ($\phi = 1$), the time-inconsistency problem does not exist and neither does the stabilization bias. We, thus, do not need to delegate to the central bank a different weight from that of society to change the trade-off between target stabilizations. Thus, $\lambda^b = \lambda$.

When the Phillips curve involves an element of expectations ($0 \leq \phi < 1$), the time-inconsistency problem and, thus, the stabilization bias exist, and the delegated weight must differ from the social weight ($\lambda^b \neq \lambda$) to make the target stabilization trade-off with discretion under the delegated loss function equal to that with commitment under the social loss function.

6.2. The delegated weight for the principally backward-looking Phillips curve ($\phi_c \le \phi < 1$)

When the Phillips curve exhibits principally backward-looking behavior ($\phi_c \le \phi < 1$), the coefficient of the weight λ^b depends only on the dynamic parameters (ϕ, β, and ρ), and does not relate to the system convergence persistence, δ. By Eq. (40),

$$\lambda^b = (\phi - \phi_c)\frac{(1-\beta\rho^2)}{\rho(1-\phi\beta\rho)}\lambda = \left[1 - \frac{(1-\phi)}{\rho(1-\phi\beta\rho)}\right]\lambda < \lambda. \tag{45}$$

Therefore, the central bank must exhibit conservatism ($\lambda^b < \lambda$).

Intuitively, if the inflation rate persists ($\phi_c \le \phi < 1$), then current inflation rate deviation from its target value will persist into future and, thus, cause losses. To reduce these losses, we must place more weight on the inflation rate stabilization (i.e., $\lambda^b < \lambda$).

6.3. The delegated weight for the principally forward-looking Phillips curve ($0 < \phi \le \phi_c$)

When the Phillips curve exhibits principally forward-looking behavior ($0 < \phi \le \phi_c$), the weight in Eq. (44) depends not only on the dynamic parameters but also on the system convergence persistence. A weight-liberal or weight-conservative central banker may emerge under different circumstances. The evaluation of the weight becomes more subtle because of forward-looking behavior. We see no analytical way to discuss the relationship of the weight with model parameters.

6.4. The delegated weight for the purely forward-looking Phillips curve ($\phi = 0$)

For the purely forward-looking case ($\phi = 0$), the delegated weight in Eq. (44) reduces to

$$\lambda^b = \frac{(1-\rho)}{\delta(1-\delta\beta\rho)}\lambda. \tag{46}$$

We report the following conditions

$$\lambda^b < \lambda \qquad \text{if } \rho > \rho_c, \tag{47a}$$

$$\lambda^b = \lambda \qquad \text{if } \rho = \rho_c, \text{ and} \tag{47b}$$

$$\lambda^b > \lambda \qquad \text{if } \rho < \rho_c, \tag{47c}$$

where $\rho_c = (1-\delta)\big/(1-\beta\delta^2)$, and $0 < \rho_c < 1$.

The central bank is weight-conservative (weight-liberal), if cost-push shock persistence is greater (less) than the critical value, ρ_c. Actually, $\partial\lambda^b/\partial\rho = -\lambda(1-\beta\delta)\big/\left[\delta(1-\delta\beta\rho)^2\right] < 0$. That is, a more persistent cost-push shock implies less weight on output stabilization. To see this, iterating the Phillips curve (Eq. 3 or Eq. 27) forward produces $\pi_t = E_t \sum_{i=0}^{\infty} \beta^i \left(\kappa x_{t+i} + u_{t+i}\right)$ or $\left(\pi_t - \hat{\pi}^{ss}\right) = E_t \sum_{i=0}^{\infty} \beta^i \left[\kappa\left(x_{t+i} - \hat{x}^{ss}\right) + u_{t+i}\right]$ with $\phi = 0$. Inflation stabilization depends entirely on current and expected future output gap stabilization and cost-push shocks, because of the purely forward-looking nature of the Phillips curve. Thus, more persistent cost-push shocks imply that more losses occur because more inflation deviation emerges, driven by current and future cost-push shocks. To reduce the losses caused by current and future inflation deviations, the policy makers must place more weight on inflation stabilization and, thus, less weight on output stabilization.

7. Summary and conclusions

In this paper, we tackle two issues in policymaking – target controllability and time inconsistency in a hybrid new-Keynesian model. Both issues do not exist in the case of an equal number of instruments and targets. They, however, emerge when fewer instruments exist than targets. Time inconsistency also results from expectations (multiple decision makers). We argue that policy makers must address target controllability before time inconsistency to avoid entangling the two issues when addressing time inconsistency. We delegate a loss function to the central bank, determining the central bank's long- and short-run target values as well as the weight parameter. A proper target value trade-off solves target controllability, whereas a proper relative weight between target stabilizations achieves the optimal target variability trade-off and solves time inconsistency.

We obtain main results as follows. First, target values are jointly asymptotically controllable if and only if they satisfy the steady-state Phillips curve. That is, controllable target values exhibit trade-offs, conform to the macroeconomic structure, and do not depend on the policy implementation style -- commitment or discretion. In other words, infinite pairs of target values can lead to joint asymptotic controllability. The efficient equilibrium corresponds to a zero target value of inflation rate. An inefficiency (monopolistic and/or tax distortions), however, requires a positive target value of inflation rate. This explains to some extent why targeting a positive inflation rate prevails in the real world. We call the steady-state Phillips curve the long-run target level trade-off equation.

Second, if desired target values do not satisfy the steady-state Phillips curve, the steady-state values of target variables exhibit regression to their respective target values. That is, the steady-state value of the inflation rate rises above its target value, when the target value of the inflation rate is low relative to that of the output gap; and vice versa. The regressive nature explains the constant average inflation bias. The constant average inflation bias, however, disappears when target values satisfy the steady-state Phillips curve. Therefore, we attribute the constant average inflation bias to target controllability (fewer instruments than targets), rather than to time inconsistency as the literature stresses.

Third, delegation parameters, which represent the trade-offs between target values and between target stabilizations, exhibit the following characteristics: (1) The central bank's long-run target values just equal the steady-state values of target variables with commitment under the social loss function. They completely depend on the model parameters, and, thus, are predetermined. They satisfy the long-run target level trade-off equation, and, thus, are jointly asymptotically controllable with discretion under the central bank loss function. They remove the constant average inflation bias, and guide monetary policy towards a correct end. (2) The short-run inflation target value conforms to the inflation behavior in Phillips curve. That is, it is lagged (expected) if the inflation rate exhibits principally backward-looking (forward-looking) behavior in the Phillips curve. Both short-run target values of the inflation rate and the output gap, though state-contingent, are predetermined and, thus, feasible in practice. With the short-run target values, the discretionary policy proves path controllable and, thus, eliminates the state-contingent inflation bias. The short-run target values further bind monetary policy to follow the optimal equilibrium paths during the evolution towards the correct end, and achieve the long-run target values step by step. (3) Delegating a proper weight to the central bank can eliminate the stabilization bias. When the inflation rate exhibits more persistence than expectation, the central bank must place more weight on inflation rate stabilization. When the inflation rate exhibits principally forward-looking behavior, evaluation of the delegated weight, however, becomes more subtle because of forward-looking behavior and the central bank may adopt a liberal or conservative weight. In any case, the delegated weight depends on dynamic parameters and the system convergence persistence.

Fourth, when the Phillips curve exhibits purely backward-looking behavior with no expectations, the time-inconsistency problem does not exist and the central bank must make the same target variability trade-offs as society desires. By contrast, When the Phillips curve involves an element of expectations, the time-inconsistency problem does exist, and the delegated weight must differ from the social weight to make the target stabilization trade-off with discretion under the delegated loss function equal to that with commitment under the social loss function.

In sum, the delegated target values are moderate and controllable, establishing monetary policy credibility. Discretionary policy under the delegated loss function, which replicates optimal policy under the social loss function, proves optimal and time consistent.

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Appendix A. Optimal solution

The optimization problem minimizes the social intertemporal loss function (1) with period loss function (2) subject to the Phillips curve (Eq. 3). Its Lagrangian expression equals the following

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \begin{Bmatrix} \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\left(x_t - x^*\right)^2\right] \\ +\psi_t\left[\kappa x_t + \phi \pi_{t-1} + (1-\phi)\beta\pi_{t+1} + u_t - \pi_t\right] \end{Bmatrix}. \tag{A.1}$$

The first-order conditions equal

$$\frac{\partial \mathcal{L}}{\partial x_t} = E_0\left\{\beta^t\left[\lambda\left(x_t - x^*\right) + \kappa\psi_t\right]\right\} = 0 \qquad \text{for } t \geq 0, \tag{A.2a}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_0} = E_0\left[\left(\pi_0 - \pi^*\right) - \psi_0 + \beta\phi\psi_1\right] = 0 \qquad \text{for } t = 0\text{, and} \tag{A.2b}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = E_0 \begin{Bmatrix} \beta^t\left[\left(\pi_t - \pi^*\right) - \psi_t\right] \\ +\beta^{t+1}\phi\psi_{t+1} + \beta^{t-1}(1-\phi)\beta\psi_{t-1} \end{Bmatrix} = 0 \qquad \text{for } t \geq 1. \tag{A.2c}$$

Eliminating the multipliers from Eqs. (A.2) gives the consolidated first-order conditions as follows

$$\left(x_0 - x^*\right) - \phi\beta\left(E_0 x_1 - x^*\right) = -\frac{\kappa}{\lambda}\left(\pi_0 - \pi^*\right) \qquad \text{for } t = 0, \tag{A.3a}$$

$$\begin{aligned}\left(E_0 x_t - x^*\right) - \phi\beta\left(E_0 x_{t+1} - x^*\right) - (1-\phi)\left(E_0 x_{t-1} - x^*\right) \\ = -\frac{\kappa}{\lambda}\left(E_0 \pi_t - \pi^*\right)\end{aligned} \qquad \text{for } t \geq 1. \tag{A.3b}$$

A.1. The solution to x_t

Combining Eqs. (A.3a) and (A.3b) with the Phillips curve (Eq. 3) produces

$$\begin{aligned} & a\beta^2 E_0 x_2 - \beta E_0 x_1 + \left[1 + \frac{\kappa^2}{\lambda} + (1-\phi)^2\beta\right]x_0 + \frac{\kappa}{\lambda}u_0 - \phi\frac{\kappa}{\lambda}\left(\pi^* - \pi_{-1}\right) \\ & -(1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] - \left[(1-\phi) + \phi^2(1-\beta)\right]x^* \\ & = 0 \qquad \text{for } t = 0, \text{ and} \end{aligned} \tag{A.4a}$$

$$a\beta^2 E_0 x_{t+2} - \beta E_0 x_{t+1} + bE_0 x_t - E_0 x_{t-1} + aE_0 x_{t-2} + \frac{\kappa}{\lambda} E_0 u_t$$
$$-(1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] = 0 \qquad \text{for } t \geq 1, \tag{A.4b}$$

where

$$a \equiv \phi(1-\phi), \tag{A.5a}$$

$$b \equiv 1 + \frac{\kappa^2}{\lambda} + (1-\phi)^2 \beta + \phi^2 \beta = 1 + \beta + \frac{\kappa^2}{\lambda} - 2a\beta, \text{ and} \tag{A.5b}$$

$$x_{-1} \equiv 0. \tag{A.5c}$$

Solve Eqs. (A.4a) and (A.4b) backwards. Assume that δ equals a root of the characteristic equation (A.4b)

$$a\beta^2\delta^4 - \beta\delta^3 + b\delta^2 - \delta + a = 0. \tag{A.6}$$

Assume that the solution of Eq. (A.4b) for $t \geq 1$ takes the following form:

$$x_t = \delta x_{t-1} + e + fu_t. \tag{A.7}$$

Using Eq. (A.7) and $E_0 u_t = \rho E_0 u_{t-1}$ leads to

$$E_0 x_t = \delta E_0 x_{t-1} + e + f\rho E_0 u_{t-1}, \tag{A.8a}$$

$$E_0 x_{t+1} = \delta^2 E_0 x_{t-1} + (\delta+1)e + (\delta+\rho) f\rho E_0 u_{t-1}, \text{ and} \tag{A.8b}$$

$$E_0 x_{t+2} = \delta^3 E_0 x_{t-1} + \left(\delta^2 + \delta + 1\right)e + \left(\delta^2 + \delta\rho + \rho^2\right) f\rho E_0 u_{t-1}. \tag{A.8c}$$

Substituting Eqs. (A.8) and $E_0 u_t = \rho E_0 u_{t-1}$ into Eq. (A.4b) results in

$$\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right)E_0 x_{t-1} + aE_0 x_{t-2}$$
$$+ce - (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] + \left(df + \frac{\kappa}{\lambda}\right)\rho E_0 u_{t-1} = 0, \text{ where} \tag{A.9}$$

$$c \equiv a\beta^2\left(\delta^2 + \delta + 1\right) - \beta(\delta+1) + b, \text{ and} \tag{A.10a}$$

$$d \equiv a\beta^2\left(\delta^2 + \delta\rho + \rho^2\right) - \beta\left(\delta + \rho\right) + b. \tag{A.10b}$$

Transforming Eq. (A.9) and noting that $-1\big/\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right) = \delta/a$ from Eq. (A.6) produces

$$\begin{aligned} E_0 x_{t-1} = \delta E_0 x_{t-2} + \frac{\delta}{a}\left\{ce - (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right]\right\} \\ + \frac{\delta}{a}\left(df + \frac{\kappa}{\lambda}\right)\rho E_0 u_{t-1}. \end{aligned} \tag{A.11}$$

Comparing Eq. (A.11) with Eq. (A.7) implies

$$e = \frac{\delta(1-\phi)(1-\beta)}{\delta c - a}\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right], \tag{A.12a}$$

$$f = -\frac{\kappa\delta\rho}{\lambda\left(\delta\rho d - a\right)}. \tag{A.12b}$$

To interpret the constant *e* in Eq. (A.7), we compute

$$\hat{x}^{ss} \equiv \frac{e}{1-\delta} = \frac{\delta(1-\phi)(1-\beta)}{(\delta c - a)(1-\delta)}\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right]. \tag{A.13}$$

With the notation $\hat{x}^{ss}$, the solution of the output gap for $t \geq 1$ equals

$$\left(x_t - \hat{x}^{ss}\right) = \delta\left(x_{t-1} - \hat{x}^{ss}\right) + fu_t, \tag{A.14}$$

where δ, $\hat{x}^{ss}$ and f are defined respectively in Eqs. (A.6), (A.13 or A.19) and (A.12b).

To obtain another form of $\hat{x}^{ss}$, compute

$$\begin{aligned} \delta c - a &= \delta\left[a\beta^2\left(\delta^2 + \delta + 1\right) - \beta\left(\delta + 1\right) + b\right] - a \\ &= \left(a\beta^2\delta^3 - \beta\delta^2 + b\delta\right) + a\beta^2\delta^2 + a\beta^2\delta - \beta\delta - a, \text{ and} \end{aligned} \tag{A.15}$$

Noting that $a\beta^2\delta^4 - \beta\delta^3 + b\delta^2 = \delta - a$ from Eq. (A.6), (A.15) becomes

$$\begin{aligned} (\delta c - a)\delta &= \left(a\beta^2\delta^4 - \beta\delta^3 + b\delta^2\right) + a\beta^2\delta^3 + a\beta^2\delta^2 - \beta\delta^2 - a\delta \\ &= \delta - a + a\beta^2\delta^3 + a\beta^2\delta^2 - \beta\delta^2 - a\delta. \end{aligned} \tag{A.16}$$

Subtracting Eq. (A.16) from Eq. (A.15) gives

$$(\delta c - a)(1-\delta) = (a\beta^2 + b - \beta - 1 + a)\delta. \tag{A.17}$$

Substituting $a\beta^2 + b - \beta - 1 + a = \frac{\kappa^2}{\lambda} + a(1-\beta)^2$ from Eq. (A.5b) into Eq. (A.17) yields

$$(\delta c - a)(1-\delta) = \left[\frac{\kappa^2}{\lambda} + a(1-\beta)^2\right]\delta. \tag{A.18}$$

Substituting Eq. (A.18) into Eq. (A.13) gives

$$\hat{x}^{ss} \equiv \frac{e}{1-\delta} = (1-\phi)(1-\beta)\left[\frac{\kappa\pi^* + \lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2}\right]. \tag{A.19}$$

Solve for x_0. Applying expectations E_0 to Eq. (A.7) for t=1 and t=2 generates

$$E_0 x_1 = \delta x_0 + e + f\rho u_0 = \delta x_0 + (1-\delta)\hat{x}^{ss} + f\rho u_0, \text{ and} \tag{A.20a}$$

$$E_0 x_2 = \delta E_0 x_1 + e + f\rho^2 u_0 = \delta^2 x_0 + (1-\delta^2)\hat{x}^{ss} + (\delta+\rho) f\rho u_0. \tag{A.20b}$$

Substituting $(1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] = \left[\frac{\kappa^2}{\lambda} + a(1-\beta)^2\right]\hat{x}^{ss}$ from Eq. (A.19), Eqs. (A.20a) and (A.20b) into Eq. (A.4a) for t=0 leads to

$$\begin{aligned}
&\left\{a\beta^2\delta^2 - \beta\delta + \left[1 + \frac{\kappa^2}{\lambda} + (1-\phi)^2\beta\right]\right\}x_0 \\
&-\left[a\beta^2\delta^2 - \beta\delta + a + \left(\beta + \frac{\kappa^2}{\lambda} - 2a\beta\right)\right]\hat{x}^{ss} - \phi\frac{\kappa}{\lambda}\left(\pi^* - \pi_{-1}\right) \\
&-\left(1 - a - \phi^2\beta\right)x^* + a\beta^2(\delta+\rho) f\rho u_0 - f\beta\rho u_0 + \frac{\kappa}{\lambda}u_0 = 0.
\end{aligned} \tag{A.21}$$

Substituting $1 + \frac{\kappa^2}{\lambda} + (1-\phi)^2\beta = b - \phi^2\beta$ and $\beta + \frac{\kappa^2}{\lambda} - 2a\beta = b - 1$ from Eq. (A.5b), $\frac{\kappa}{\lambda} = -f\left(d - \frac{a}{\delta\rho}\right)$ from Eq. (A.12b) and d in Eq. (A.10b) into Eq. (A.21) generates

$$\left(a\beta^2\delta^2-\beta\delta+b-\phi^2\beta\right)\left(x_0-\hat{x}^{ss}\right)=\left(1-a-\phi^2\beta\right)\left(x^*-\hat{x}^{ss}\right)+\phi\frac{\kappa}{\lambda}\left(\pi^*-\pi_{-1}\right)+\left(a\beta^2\delta^2-\beta\delta+b-\frac{a}{\delta\rho}\right)fu_0. \quad \text{(A.22)}$$

Denote

$$\delta_{x0}=\frac{\left(1-a-\phi^2\beta\right)}{\left(a\beta^2\delta^2-\beta\delta+b-\phi^2\beta\right)}, \text{ and} \quad \text{(A.23a)}$$

$$f_0\equiv\frac{\left(a\beta^2\delta^2-\beta\delta+b-\frac{a}{\delta\rho}\right)f}{\left(a\beta^2\delta^2-\beta\delta+b-\phi^2\beta\right)}=\frac{-\kappa\left[\left(a\beta^2\delta^2-\beta\delta+b\right)\delta\rho-a\right]}{\lambda\left(a\beta^2\delta^2-\beta\delta+b-\phi^2\beta\right)\left(\delta\rho d-a\right)}. \quad \text{(A.23b)}$$

With the notations, Eq. (A.22) becomes

$$\left(x_0-\hat{x}^{ss}\right)=\delta_{x0}\left[\left(x^*-\hat{x}^{ss}\right)+\frac{\phi\kappa\left(\pi^*-\pi_{-1}\right)}{\lambda\left(1-a-\phi^2\beta\right)}\right]+f_0u_0. \quad \text{(A.24)}$$

Note that $\delta_{x0}=\delta$, $f_0=f$, and Eq. (A.24) reduces to $\left(x_0-\hat{x}^{ss}\right)=\delta\left(x^*-\hat{x}^{ss}\right)+fu_0$, when ϕ=0.

A.2. The solution to π_t

Combining Eqs. (A.3a) and (A.3b) with the Phillips curve (Eq. 34) and using $E_0u_{t+1}=\rho E_0u_t$ generates

$$a\beta^2E_0\pi_2-\beta E_0\pi_1+\left(1+\frac{\kappa^2}{\lambda}+\phi^2\beta\right)\pi_0-\left(1-\phi\beta\rho\right)u_0-\kappa\left[\frac{\kappa}{\lambda}\pi^*+\phi\left(1-\beta\right)x^*\right]-\phi\pi_{-1}-\left(1-\phi\right)\kappa x^*=0 \quad \text{for } t=0\text{, and} \quad \text{(A.25a)}$$

$$a\beta^2E_0\pi_{t+2}-\beta E_0\pi_{t+1}+bE_0\pi_t-E_0\pi_{t-1}+aE_0\pi_{t-2}-\kappa\left[\frac{\kappa}{\lambda}\pi^*+\phi\left(1-\beta\right)x^*\right]-\left(1-\phi\beta\rho\right)E_0u_t+\left(1-\phi\right)E_0u_{t-1}=0 \quad \text{for } t\geq 1, \quad \text{(A.25b)}$$

where *a* and *b* are defined, respectively, in Eqs. (A.5a) and (A.5b).

Solve Eqs. (A.25a) and (A.25b) backwards. The characteristic equation of Eq. (A.25b) equals Eq. (A.6). Assume that the solution of Eq. (A.25b) takes the following form

$$\pi_t = \delta\pi_{t-1} + g + h\left[\left(1-\phi\beta\rho\right)u_t - \left(1-\phi\right)u_{t-1}\right], \qquad \text{(A.26)}$$

where δ is a root of the characteristic equation (A.6).

Using Eq. (A.26) and $E_0 u_t = \rho E_0 u_{t-1}$ leads to

$$E_0\pi_t = \delta E_0\pi_{t-1} + g + h\rho\left[\left(1-\phi\beta\rho\right)E_0u_{t-1} - \left(1-\phi\right)E_0u_{t-2}\right], \qquad \text{(A.27a)}$$

$$E_0\pi_{t+1} = \delta^2 E_0\pi_{t-1} + \left(\delta+1\right)g + \left(\delta+\rho\right)h\rho\left[\left(1-\phi\beta\rho\right)E_0u_{t-1} - \left(1-\phi\right)E_0u_{t-2}\right], \qquad \text{(A.27b)}$$

$$E_0\pi_{t+2} = \delta^3 E_0\pi_{t-1} + \left(\delta^2+\delta+1\right)g + \left(\delta^2+\delta\rho+\rho^2\right)h\rho\left[\left(1-\phi\beta\rho\right)E_0u_{t-1} - \left(1-\phi\right)E_0u_{t-2}\right]. \qquad \text{(A.27c)}$$

Substituting Eqs. (A.27a) to (A.27c), and $E_0 u_t = \rho E_0 u_{t-1}$ into Eq. (A.25b) results in

$$\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right)E_0\pi_{t-1} + aE_0\pi_{t-2} + cg - \kappa\left[\frac{\kappa}{\lambda}\pi^* + \phi\left(1-\beta\right)x^*\right] + \left(dh-1\right)\rho\left[\left(1-\phi\beta\rho\right)E_0u_{t-1} - \left(1-\phi\right)E_0u_{t-2}\right] = 0, \qquad \text{(A.28)}$$

where c and d are defined respectively in Eqs. (A.10a) and (A.10b).

Transforming Eq. (A.28) and noting that $-1/\left(a\beta^2\delta^3 - \beta\delta^2 + b\delta - 1\right) = \delta/a$ from Eq. (A.6) produces

$$E_0\pi_{t-1} = \delta E_0\pi_{t-2} + \frac{\delta}{a}\left\{cg - \kappa\left[\frac{\kappa}{\lambda}\pi^* + \phi\left(1-\beta\right)x^*\right]\right\} + \frac{\delta}{a}\left(dh-1\right)\rho\left[\left(1-\phi\beta\rho\right)E_0u_{t-1} - \left(1-\phi\right)E_0u_{t-2}\right]. \qquad \text{(A.29)}$$

Comparing Eq. (A.29) with Eq. (A.26) implies

$$g = \frac{\kappa\delta}{\delta c - a}\left[\frac{\kappa}{\lambda}\pi^* + \phi\left(1-\beta\right)x^*\right], \text{ and} \qquad \text{(A.30a)}$$

$$h = \frac{\delta\rho}{\delta\rho d - a}. \qquad \text{(A.30b)}$$

Substituting $\frac{\delta}{(\delta c - a)} = \frac{\lambda(1-\delta)}{\kappa^2 + \lambda a(1-\beta)^2}$ from Eq. (A.18) into Eq. (A.30a) produces

$$g = \kappa(1-\delta)\left[\frac{\kappa\pi^* + \lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2}\right]. \tag{A.31}$$

Denote

$$\hat{\pi}^{ss} \equiv \frac{g}{1-\delta} = \kappa\left[\frac{\kappa\pi^* + \lambda\phi(1-\beta)x^*}{\kappa^2 + \lambda a(1-\beta)^2}\right]. \tag{A.32}$$

Using the AR(1) process of shocks, $u_t = \rho u_{t-1} + \hat{u}_t$, the term that involves shocks in Eq. (A.26) produces

$$(1-\phi\beta\rho)u_t - (1-\phi)u_{t-1} = (\phi - \phi_c)(1-\beta\rho^2)u_{t-1} + (1-\phi\beta\rho)\hat{u}_t \text{, where} \tag{A.33}$$

$$\phi_c = \frac{1-\rho}{1-\beta\rho^2} \text{, and } 0 < \phi_c \leq 1. \tag{A.34}$$

With the notation $\hat{\pi}^{ss}$, the solution of the inflation in Eq. (A.26) for $t \geq 1$ equals

$$\begin{aligned}(\pi_t - \hat{\pi}^{ss}) &= \delta(\pi_{t-1} - \hat{\pi}^{ss}) + h\left[(1-\phi\beta\rho)u_t - (1-\phi)u_{t-1}\right], \quad \text{or} \\ (\pi_t - \hat{\pi}^{ss}) &= \delta(\pi_{t-1} - \hat{\pi}^{ss}) + h\left[(\phi-\phi_c)(1-\beta\rho^2)u_{t-1} + (1-\phi\beta\rho)\hat{u}_t\right],\end{aligned} \tag{A.35}$$

where δ, $\hat{\pi}^{ss}$, ϕ_c, and h are defined respectively in Eqs. (A.6), (A.32), (A.34) and (A.30b).

Solve for π_0. Applying expectations E_0 to Eq. (A.35) for t=1 and t=2 generates

$$E_0\pi_1 = \delta\pi_0 + (1-\delta)\hat{\pi}^{ss} + h(\phi-\phi_c)(1-\beta\rho^2)u_0 \text{, and} \tag{A.36a}$$

$$E_0\pi_2 = \delta^2\pi_0 + (1-\delta^2)\hat{\pi}^{ss} + (\delta+\rho)h(\phi-\phi_c)(1-\beta\rho^2)u_0. \tag{A.36b}$$

Substituting $\kappa\left[\frac{\kappa}{\lambda}\pi^* + \phi(1-\beta)x^*\right] = \left[\frac{\kappa^2}{\lambda} + a(1-\beta)^2\right]\hat{\pi}^{ss}$ from Eq. (A.32), Eqs. (A.36a) and (A.36b) into Eq. (A.25a) for t=0 and arranging leads to

$$\left[a\beta^2\delta^2-\beta\delta+\left(1+\frac{\kappa^2}{\lambda}+\phi^2\beta\right)\right]\pi_0-\left[\begin{array}{l}a\beta^2\delta^2-\beta\delta+a\\ \qquad+\left(\beta+\frac{\kappa^2}{\lambda}-2a\beta\right)\end{array}\right]\hat{\pi}^{ss}-\phi\pi_{-1} \quad \text{(A.37)}$$
$$-(1-\phi)\kappa x^*-\left\{\beta\left[1-a\beta(\delta+\rho)\right]h(\phi-\phi_c)\left(1-\beta\rho^2\right)+(1-\phi\beta\rho)\right\}u_0=0.$$

Substituting $1+\frac{\kappa^2}{\lambda}+\phi^2\beta=b-(1-\phi)^2\beta$ and $\beta+\frac{\kappa^2}{\lambda}-2a\beta=b-1$ from Eq. (A.5b) into Eq. (A.37) yields

$$\left[a\beta^2\delta^2-\beta\delta+b-(1-\phi)^2\beta\right]\left(\pi_0-\hat{\pi}^{ss}\right)+(1-a)\hat{\pi}^{ss}-(1-\phi)(1-\phi)\beta\hat{\pi}^{ss}$$
$$-\phi\pi_{-1}-(1-\phi)\kappa x^*-\left\{\begin{array}{c}\beta\left[1-a\beta(\delta+\rho)\right]h(\phi-\phi_c)\left(1-\beta\rho^2\right)\\ +(1-\phi\beta\rho)\end{array}\right\}u_0=0. \quad \text{(A.38)}$$

From Eqs. (A.19) and (A.32), $\kappa\hat{x}^{ss}=(1-\phi)(1-\beta)\hat{\pi}^{ss}$. Thus, $(1-\phi)\beta\hat{\pi}^{ss}=(1-\phi)\hat{\pi}^{ss}-\kappa\hat{x}^{ss}$.

Substituting it into Eq. (A.38) produces

$$\left(\pi_0-\hat{\pi}^{ss}\right)=\delta_{\pi 0}\left[\phi\left(\pi_{-1}-\hat{\pi}^{ss}\right)+(1-\phi)\kappa\left(x^*-\hat{x}^{ss}\right)\right]+h_0u_0, \text{ where} \quad \text{(A.39)}$$

$$\delta_{\pi 0}\equiv\frac{1}{a\beta^2\delta^2-\beta\delta+b-(1-\phi)^2\beta}, \text{ and} \quad \text{(A.40a)}$$

$$h_0\equiv\frac{\beta\left[1-a\beta(\delta+\rho)\right]h(\phi-\phi_c)\left(1-\beta\rho^2\right)+(1-\phi\beta\rho)}{a\beta^2\delta^2-\beta\delta+b-(1-\phi)^2\beta}. \quad \text{(A.40b)}$$

Note that $\delta_{x0}=\delta$ and Eq. (A.39) reduces to $\left(\pi_0-\hat{\pi}^{ss}\right)=\delta\left(\pi_{-1}-\hat{\pi}^{ss}\right)+h_0u_0$, when ϕ=1.

Appendix B. Discretionary policy with social target values, π^* and x^*

Under discretionary policy the central bank re-formulates policy each period, so what is relevant is the consolidated first-order condition for t=0, Eq. (A.3a). We rewrite Eq. (A.3a) here with subscript t instead of subscript 0

$$\left(x_t-x^*\right)-\phi\beta\left(E_tx_{t+1}-x^*\right)=-\frac{\kappa}{\lambda}\left(\pi_t-\pi^*\right) \quad \text{(B.1)}$$

B.1. The solution to x_t

Using the first-order condition (B.1) and the Phillips curve (Eq. 3) obtains discretionary policy. Specifically, transforming Eq. (B.1), Lagging and leading one period produces

$$\frac{\kappa}{\lambda}\pi_t = -x_t + \phi\beta E_t x_{t+1} + \frac{\kappa}{\lambda}\pi^* + (1-\phi\beta)x^* \tag{B.2a}$$

$$\frac{\kappa}{\lambda}\pi_{t-1} = -x_{t-1} + \phi\beta E_{t-1} x_t + \frac{\kappa}{\lambda}\pi^* + (1-\phi\beta)x^* \tag{B.2b}$$

$$\frac{\kappa}{\lambda}\pi_{t+1} = -x_{t+1} + \phi\beta E_{t+1} x_{t+2} + \frac{\kappa}{\lambda}\pi^* + (1-\phi\beta)x^* \tag{B.2c}$$

Substituting Eqs. (B.2) into the Phillips curve (Eq. 3), which is multiplied by κ/λ, and arranging terms produces

$$a\beta^2 E_t x_{t+2} - \beta E_t x_{t+1} + b' x_t - \phi x_{t-1} - (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + (1-\phi\beta)x^*\right] + \frac{\kappa}{\lambda}u_t - \phi^2\beta\left(x_t - E_{t-1}x_t\right) = 0, \quad \text{where} \tag{B.3}$$

$$b' = 1 + \frac{\kappa^2}{\lambda} + \phi^2\beta\,. \tag{B.4}$$

Assume that the solution of Eq. (B.3) takes the form

$$x_t = \bar{\delta} x_{t-1} + \bar{e} + \bar{f} u_t + \underline{f} u_{t-1}. \tag{B.5}$$

Using Eq. (B.5) generates

$$E_t x_{t+1} = \bar{\delta} x_t + \bar{e} + \bar{f}\rho u_t + \underline{f} u_t\,, \tag{B.6a}$$

$$E_t x_{t+2} = \bar{\delta}^2 x_t + \left(1+\bar{\delta}\right)\bar{e} + \left(\bar{\delta}+\rho\right)\bar{f}\rho u_t + \left(\bar{\delta}+\rho\right)\underline{f} u_t\,, \text{ and} \tag{B.6b}$$

$$x_t - E_{t-1}x_t = \bar{f}\left(u_t - \rho u_{t-1}\right). \tag{B.6c}$$

Substituting Eqs. (B.6) into (B.3) produces

$$x_t = \frac{\phi}{a\beta^2\bar{\delta}^2 - \beta\bar{\delta} + b'} x_{t-1} + \frac{\beta\left[1 - a\beta\left(1+\bar{\delta}\right)\right]\bar{e} + (1-\phi)(1-\beta)\left[\frac{\kappa}{\lambda}\pi^* + (1-\phi\beta)x^*\right]}{a\beta^2\bar{\delta}^2 - \beta\bar{\delta} + b'}$$
$$+\left\{\frac{\left\{\rho\left[1 - a\beta\left(\bar{\delta}+\rho\right)\right] + \phi^2\right\}\beta\bar{f} + \left[1 - a\beta\left(\bar{\delta}+\rho\right)\right]\beta\underline{f} - \frac{\kappa}{\lambda}}{a\beta^2\bar{\delta}^2 - \beta\bar{\delta} + b'}\right\}u_t - \frac{\phi^2\beta\rho\bar{f}}{a\beta^2\bar{\delta}^2 - \beta\bar{\delta} + b'}u_{t-1}. \quad \text{(B.7)}$$

Comparing Eq. (B.7) with Eq. (B.5) produces

$$a\beta^2\bar{\delta}^2 - \beta\bar{\delta} + b' = \frac{\phi}{\bar{\delta}}, \quad \text{(B.8a)}$$

$$\bar{e} = (1-\phi)(1-\beta)\left\{\frac{\kappa\pi^*/\lambda + (1-\phi\beta)x^*}{\phi/\bar{\delta} - \beta\left[1 - a\beta\left(1+\bar{\delta}\right)\right]}\right\}, \quad \text{(B.8b)}$$

$$\bar{f} = -\frac{\kappa}{\lambda\left(1-\phi\bar{\delta}\beta\right)\left\{\phi/\bar{\delta} - \beta\rho\left[1 - a\beta\left(\bar{\delta}+\rho\right)\right]\right\}}, \text{ and} \quad \text{(B.8c)}$$

$$\underline{f} = -\bar{\delta}\phi\beta\rho\bar{f}. \quad \text{(B.8d)}$$

To interpret the constant $\bar{e}$ in Eq. (B.5), we denote

$$\bar{x}^{ss} \equiv \frac{\bar{e}}{1-\bar{\delta}} = (1-\phi)(1-\beta)\left\{\frac{\kappa\pi^*/\lambda + (1-\phi\beta)x^*}{\left(1-\bar{\delta}\right)\left\{\phi/\bar{\delta} - \beta\left[1 - a\beta\left(1+\bar{\delta}\right)\right]\right\}}\right\} \quad \text{(B.9)}$$

With the notation $\bar{x}^{ss}$, the solution of the output gap equals

$$x_t - \bar{x}^{ss} = \bar{\delta}\left(x_{t-1} - \bar{x}^{ss}\right) + \bar{f}\left(u_t - \bar{\delta}\phi\beta\rho u_{t-1}\right) \quad \text{(B.10)}$$

To obtain another expression of $\bar{x}^{ss}$ without $\bar{\delta}$, compute the denominator in the brace. Using Eq. (B.8a), arranging terms, using Eq. (B.8a) again, and definitions of a and b' in sequence produces

$$
\begin{aligned}
&\left(1-\bar{\delta}\right)\left\{\phi/\bar{\delta}-\beta\left[1-a\beta\left(1+\bar{\delta}\right)\right]\right\} \\
&=\left(1-\bar{\delta}\right)\left\{a\beta^2\bar{\delta}^2-\beta\bar{\delta}+b'-\beta\left[1-a\beta\left(1+\bar{\delta}\right)\right]\right\} \\
&=b'-\beta+a\beta^2-\bar{\delta}\left(a\beta^2\bar{\delta}^2-\beta\bar{\delta}+b'\right) \\
&=b'-\beta+a\beta^2-\phi \\
&=\kappa^2/\lambda+\left(1-\phi\right)\left(1-\beta\right)\left(1-\phi\beta\right)
\end{aligned} \tag{B.11}
$$

Substituting Eq. (B.11) into Eq. (B.9) gives

$$
\bar{x}^{ss}\equiv\frac{\bar{e}}{1-\bar{\delta}}=\left(1-\phi\right)\left(1-\beta\right)\left[\frac{\kappa\pi^*+\lambda\left(1-\phi\beta\right)x^*}{\kappa^2+\lambda\left(1-\phi\right)\left(1-\beta\right)\left(1-\phi\beta\right)}\right] \tag{B.12}
$$

B.2. The solution to π_t

The difference between Eqs. (B.2a) and (B.2b), which is multiplied by $\bar{\delta}$, equals

$$
\begin{aligned}
\frac{\kappa}{\lambda}\left(\pi_t-\bar{\delta}\pi_{t-1}\right)=-\left(x_t-\bar{\delta}x_{t-1}\right)+\phi\beta\left(E_t x_{t+1}-\bar{\delta}E_{t-1}x_t\right) \\
+\left(1-\bar{\delta}\right)\left[\frac{\kappa}{\lambda}\pi^*+\left(1-\phi\beta\right)x^*\right].
\end{aligned} \tag{B.13}
$$

Using Eq. (B.5) and $\underline{f}=-\bar{\delta}\phi\beta\rho\bar{f}$, Eq. (B.13) becomes

$$
\begin{aligned}
\pi_t=\bar{\delta}\pi_{t-1}-\left(1-\phi\beta\right)\frac{\lambda}{\kappa}\bar{e}+\frac{\lambda}{\kappa}\left(1-\bar{\delta}\right)\left[\frac{\kappa}{\lambda}\pi^*+\left(1-\phi\beta\right)x^*\right] \\
-\frac{\lambda}{\kappa}\left(1-\bar{\delta}\phi\beta\right)\left(1-\phi\beta\rho\right)\bar{f}u_t.
\end{aligned} \tag{B.14}
$$

Denote

$$
\begin{aligned}
\bar{g}&\equiv-\left(1-\phi\beta\right)\frac{\lambda}{\kappa}\bar{e}+\frac{\lambda}{\kappa}\left(1-\bar{\delta}\right)\left[\frac{\kappa}{\lambda}\pi^*+\left(1-\phi\beta\right)x^*\right] \\
&=\kappa\left(1-\bar{\delta}\right)\left\{\frac{\kappa\pi^*+\lambda\left(1-\phi\beta\right)x^*}{\kappa^2+\lambda\left(1-\phi\right)\left(1-\beta\right)\left(1-\phi\beta\right)}\right\},\quad\text{and}
\end{aligned} \tag{B.15a}
$$

$$
\bar{h}\equiv-\frac{\lambda}{\kappa}\left(1-\bar{\delta}\phi\beta\right)\left(1-\phi\beta\rho\right)\bar{f}=\frac{1-\phi\beta\rho}{\phi/\bar{\delta}-\beta\rho\left[1-a\beta\left(\bar{\delta}+\rho\right)\right]}. \tag{B.15b}
$$

With the notations, Eq. (B.14) equals

$$\pi_t = \bar{\delta}\pi_{t-1} + \bar{g} + \bar{h}u_t . \quad \text{(B.16)}$$

To interpret the constant $\bar{g}$ in Eq. (B.16), denote

$$\bar{\pi}^{ss} \equiv \frac{\bar{g}}{1-\bar{\delta}} = \kappa\left[\frac{\kappa\pi^* + \lambda(1-\phi\beta)x^*}{\kappa^2 + \lambda(1-\phi)(1-\beta)(1-\phi\beta)}\right]. \quad \text{(B.17)}$$

With the notation $\bar{\pi}^{ss}$, the solution of the inflation rate equals

$$\pi_t - \bar{\pi}^{ss} = \bar{\delta}\left(\pi_{t-1} - \bar{\pi}^{ss}\right) + \bar{h}u_t . \quad \text{(B.18)}$$

Combining Eqs. (B.12) and (B.17) produces

$$\kappa\bar{x}^{ss} = (1-\phi)(1-\beta)\bar{\pi}^{ss} . \quad \text{(B.19)}$$

Appendix C. Discretionary policy with long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$

C.1. Discretionary policy with long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$

The central bank operates with discretion, and always re-minimizes each period the expectation of the intertemporal loss function $\mathbb{L}^b = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t^b\right)$ with the period loss function in Eq. (25), subject to the Phillips curve around the optimal steady state in Eq. (27). The solution equals[18]

$$\left(\pi_t - \hat{\pi}^{ss}\right) = \tilde{\delta}\left(\pi_{t-1} - \hat{\pi}^{ss}\right) + \tilde{h}u_t , \text{ and} \quad \text{(C.1a)}$$

$$\left(x_t - \hat{x}^{ss}\right) = \tilde{\delta}\left(x_{t-1} - \hat{x}^{ss}\right) + \tilde{f}\left(u_t - \tilde{\delta}\phi\beta\rho u_{t-1}\right), \quad \text{(C.1b)}$$

where $\tilde{\delta}$ is a root of the characteristic equation

$$a\beta^2\tilde{\delta}^3 - \beta\tilde{\delta}^2 + \tilde{b}'\tilde{\delta} = \phi , \quad \text{(C.2a)}$$

$$\tilde{b}' \equiv 1 + \frac{\kappa^2}{\lambda^b} + \phi^2\beta , \quad \text{(C.2b)}$$

18. The computation and the form of solution are exactly the same as those for discretionary policy with social target values, π^* and x^*, in Appendix B.

$$\tilde{h} = \frac{1-\phi\beta\rho}{\phi/\tilde{\delta} - \beta\rho\left[1-a\beta\left(\tilde{\delta}+\rho\right)\right]}, \text{ and} \quad \text{(C.2c)}$$

$$\tilde{f} = -\frac{\kappa}{\lambda^b\left(1-\phi\beta\tilde{\delta}\right)\left\{\phi/\tilde{\delta} - \beta\rho\left[1-a\beta\left(\tilde{\delta}+\rho\right)\right]\right\}}. \quad \text{(C.2d)}$$

C.2. The condition for path controllability with the long-run target values, $\hat{\pi}^{ss}$ *and* $\hat{x}^{ss}$

Consider whether discretionary policy with the long-run target values, $\hat{\pi}^{ss}$ and $\hat{x}^{ss}$, can achieve path controllability (i.e., $\tilde{\delta} = \delta$). Now, derive the conditions for $\tilde{\delta} = \delta$, using the two characteristic equations, Eqs. (C.2a) and (7a), under both discretionary policy and optimal policy. When $\tilde{\delta} = \delta$, the characteristic equation (Eq. C.2a) becomes

$$a\beta^2\delta^3 - \beta\delta^2 + \left(1 + \frac{\kappa^2}{\lambda^b} + \phi^2\beta\right)\delta = \phi. \quad \text{(C.3)}$$

After substitution from Eqs. (6b) and (6c), the characteristic equation (6a) equals

$$\left[a\beta^2\delta^3 - \beta\delta^2 + \left(1 + \frac{\kappa^2}{\lambda} + \phi^2\beta\right)\delta\right]\delta + \left(1-\phi\right)^2\beta\delta^2 - \delta + \phi\left(1-\phi\right) = 0. \quad \text{(C.4)}$$

Assuming

$$\lambda^b = \lambda \quad \text{(C.5)}$$

and substituting Eq. (C.3) into Eq. (C.4) and arranging gives

$$\left(1-\phi\right)\left[\left(1-\phi\right)\beta\delta^2 - \delta + \phi\right] = 0. \quad \text{(C.6)}$$

Eq. (C.6) gives rise to the conditions for $\tilde{\delta} = \delta$ and $\lambda^b = \lambda$

$$\phi = 1, \text{ and/or} \quad \text{(C.7)}$$

$$\phi\left(1-\beta\delta^2\right) = \delta - \beta\delta^2. \quad \text{(C.8)}$$

Now, prove that $\phi\left(1-\beta\delta^2\right) \neq \delta - \beta\delta^2$ for $\phi \in (0,1)$ by contradiction. Assume Eq. (C.8) holds. Substituting Eqs. (6b) and (6c) into Eq. (6a) and arranging produces

$$\phi(1-\phi)\left(1-\beta\delta^2\right)^2+\delta\left(\delta-\beta\delta^2\right)-\left(\delta-\beta\delta^2\right)+\frac{\kappa^2}{\lambda}\delta^2=0. \tag{C.9}$$

Substituting Eq. (C.8) into Eq. (C.9) and arranging produces

$$\left(\delta-\beta\delta^2\right)\left[\delta-\beta\delta^2-\phi\left(1-\beta\delta^2\right)\right]+\frac{\kappa^2}{\lambda}\delta^2=0. \tag{C.10}$$

Using Eq. (C.8), Eq. (C.10) reduces to $\kappa^2\delta^2/\lambda=0$. Thus, $\delta=0$. With δ=0 the characteristic equation (6a) reduces to a=0. a=0 contradicts with $\phi\in(0,1)$. Therefore, $\phi\left(1-\beta\delta^2\right)\neq\delta-\beta\delta^2$ for $\phi\in(0,1)$.

In sum, if $\lambda^b=\lambda$, then

$$\tilde{\delta}=\delta\Leftrightarrow\phi=1 \tag{C.11}$$

We proved that $\tilde{\delta}=\delta\Rightarrow\phi=1$. We can easily show that $\phi=1\Rightarrow\tilde{\delta}=\delta$ by noting that Eq. (C.2a) equals Eq. (6a) when λ^b=λ and ϕ=1. Eq. (C.11) means that discretionary policy with long-run target values $\pi^b=\hat{\pi}^{ss}$ and $x^b=\hat{x}^{ss}$ as well as the weight λ^b=λ proves path controllable, when ϕ = 1.

C.3. Disappearance of the stabilization bias when ϕ = 1

Now, we prove that the stabilization bias does not exist when ϕ = 1 under discretionary policy with long-run target values $\pi^b=\hat{\pi}^{ss}$ and $x^b=\hat{x}^{ss}$ as well as the weight $\lambda^b=\lambda$.

With λ^b=λ, ϕ = 1, and $\tilde{\delta}=\delta$, Eq. (C.2c) become $\tilde{h}=\delta(1-\beta\rho)/(1-\delta\beta\rho)$. Thus,

$$\tilde{\pi}_t^u=\tilde{h}u_t=\frac{\delta(1-\beta\rho)}{(1-\delta\beta\rho)}u_t. \tag{C.12}$$

With ϕ = 1, Eq. (10a) reduces to

$$\hat{\pi}_t^u=\frac{1}{d}(1-\beta\rho)u_t=\frac{\delta(1-\beta\rho)}{(1-\delta\beta\rho)}u_t. \tag{C.13}$$

Comparing Eq. (C.12) with Eq. (C.13) yields

$$\tilde{\pi}_t^u = \hat{\pi}_t^u \,. \tag{C.14}$$

Appendix D. Discretionary policy with lagged, short-run target values in the new-Keynesian model

The central bank operates with discretion, and always re-minimizes each period, subject to the Phillips curve around the optimal steady state in Eq. (27), the expectation of the intertemporal loss function $\mathbb{L}^b = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t^b\right)$ with the period loss function in Eq. (36).

The Lagrangian expression of the problem equals

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} \dfrac{1}{2}\left\{ \begin{array}{l} \left[\left(\pi_t - \hat{\pi}^{ss}\right) - \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right)\right]^2 \\ +\lambda^b\left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right]^2 \end{array} \right\} \\ +\psi_t \left[\begin{array}{l} \kappa\left(x_t - \hat{x}^{ss}\right) + \phi\left(\pi_{t-1} - \hat{\pi}^{ss}\right) \\ +\left(1-\phi\right)\beta\left(\pi_{t+1} - \hat{\pi}^{ss}\right) + u_t - \left(\pi_t - \hat{\pi}^{ss}\right) \end{array} \right] \end{array} \right\}. \tag{D.1}$$

As the central bank re-formulates policy each period, we only calculate the first-order conditions with respect to π_0, x_0, and x_1, and eliminate the multipliers ψ_0 and ψ_1 leads to the consolidated first-order condition, replacing period 0 with period t,

$$\begin{array}{l} \kappa\left\{ \begin{array}{l} \left[\left(\pi_t - \hat{\pi}^{ss}\right) - \delta\left(\pi_{t-1} - \hat{\pi}^{ss}\right)\right] \\ -\beta\delta\left[\left(E_t\pi_{t+1} - \hat{\pi}^{ss}\right) - \delta\left(\pi_t - \hat{\pi}^{ss}\right)\right] \end{array} \right\} \\ \qquad + \lambda^b \left\{ \begin{array}{l} \left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right] \\ -\beta\left(\phi + \delta\right)\left[\left(E_t x_{t+1} - \hat{x}^{ss}\right) - \delta\left(x_t - \hat{x}^{ss}\right)\right] \\ +\phi\beta^2\delta\left[\left(E_t x_{t+2} - \hat{x}^{ss}\right) - \delta\left(E_t x_{t+1} - \hat{x}^{ss}\right)\right] \end{array} \right\} = 0. \end{array} \tag{D.2}$$

Appendix E. Discretionary policy with expected, short-run inflation target value in the new-Keynesian model

The central bank operates with discretion, and always re-minimizes each period, subject to the

Phillips curve around the optimal steady state in Eq. (27), the expectation of the intertemporal loss function $\mathbb{L}^b = E_0\left(\sum_{t=0}^{\infty}\beta^t L_t^b\right)$ with the period loss function in Eq. (42).

The Lagrangian expression of the problem equals

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} \frac{1}{2}\left\{ \begin{array}{l} \left[\left(\pi_t - \hat{\pi}^{ss}\right) - \delta^{-1}\left(\pi_{t+1} - \hat{\pi}^{ss}\right)\right]^2 \\ +\lambda^b \left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right]^2 \end{array} \right\} \\ +\psi_t \left[\begin{array}{l} \kappa\left(x_t - \hat{x}^{ss}\right) + \phi\left(\pi_{t-1} - \hat{\pi}^{ss}\right) \\ +\left(1-\phi\right)\beta\left(\pi_{t+1} - \hat{\pi}^{ss}\right) + u_t - \left(\pi_t - \hat{\pi}^{ss}\right) \end{array} \right] \end{array} \right\}. \qquad \text{(E.1)}$$

As the central bank re-makes policy each period, we only calculate the first-order conditions with respect to π_0, x_0, and x_1, and eliminate the multipliers ψ_0 and ψ_1 leads to the consolidated first-order condition, replacing period 0 with period t and multiplying the condition by δ,

$$\kappa\left[\left(E_t\pi_{t+1} - \hat{\pi}^{ss}\right) - \delta\left(\pi_t - \hat{\pi}^{ss}\right)\right] = \lambda^b \delta \left\{ \begin{array}{l} \left[\left(x_t - \hat{x}^{ss}\right) - \delta\left(x_{t-1} - \hat{x}^{ss}\right)\right] \\ -\beta\left(\delta + \phi\right)\left[\left(E_t x_{t+1} - \hat{x}^{ss}\right) - \delta\left(x_t - \hat{x}^{ss}\right)\right] \\ +\phi\delta\beta^2\left[\left(E_t x_{t+2} - \hat{x}^{ss}\right) - \delta\left(E_t x_{t+1} - \hat{x}^{ss}\right)\right] \end{array} \right\}. \qquad \text{(E.2)}$$